NUMERICAL SOLUTIONS OF SINGULARLY PERTURBED INITIAL AND BOUNDARY VALUE PROBLEMS USING SPLINES

by

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RAJESH KUMAR BAWA

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Dedicated to

My Parents

CERTIFICATE

This is to certify that the research work contained in the thesis entitled *'Numerical Solutions of Singularly Perturbed Initial and Boundary Value Problems using splines'* by *Rajesh Kumar Bawa* has been carried out under my supervision and that this work has not been submitted elsewhere for the award of any degree or diploma.

M. K. Kadalbajoo 20.7.1993

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At last but not least I make an effort to express my feelings in the form of words to my wife, Seema, without whose deep sense of understanding and patience, work would have not taken shape.

Rajesh K. Bawa

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LIST OF PUBLICATIONS

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1. Numerical Solution of Singularly Perturbed Initial Value Problems Using Fourth Order Cubic Spline Method,

   International Journal of Computer Mathematics, 46(1992), pp. 87-95.

2. Cubic Spline Method for a Class of Non-Linear Singularly Perturbed Boundary Value Problems,

   Journal of Optimization Theory and Applications, to appear.

3. Third Order, Variable-Mesh, Cubic Spline Methods for Singularly-Perturbed Boundary Value Problems,

   Applied Mathematics and Computation, to appear.

4. Third Order, Variable-Mesh, Cubic Spline Methods for Non Linear Singularly-Perturbed Boundary Value Problems, *Journal of Optimization Theory and Applications*, to appear.

5. Variable Mesh Difference Scheme for Singularly Perturbed Boundary Value Problems using Splines, submitted.

SYNOPSIS

Singular Perturbation, now is a maturing mathematical subject with a fairly long history and a strong promise for continued important applications throughout science and engineering. A Singular Perturbation problem is well defined as one in which no single asymptotic expansion is uniformly valid throughout the interval, as the perturbation parameter $\varepsilon \longrightarrow 0$. The singular perturbation problems find place in many area of engineering and applied mathematics, for instance: fluid mechanics, fluid dynamics, quantum mechanics, plasticity, chemical reactor theory and many other problems of fluid motion. A few notable examples are boundary layer and WKB problems. The problems to be solved by means of asymptotic and numerical analysis are becoming progressively more and more complicated.

The aim of this thesis is to derive some simple and efficient numerical methods using splines for solving singular perturbation problems which are easy to implement and are not costly in terms of computer time also.

Chapter I of the thesis contains general introduction which includes a brief survey of asymptotic and numerical analysis of Singular perturbation problems. A summary of recent methods and that of the present thesis is presented.

In chapter II, we first present a numerical method namely Method 1, for Linear Singularly Perturbed Initial Value Problems of the form:

$$\varepsilon y' + p(x,\varepsilon)y = q(x,\varepsilon);\ y(a) = \alpha, \quad a \le x \le b,$$

where ε is a small positive parameter, $0 < \varepsilon \ll 1$, α is a given constant and $p(x,\varepsilon) > 0$ and $q(x,\varepsilon)$ are sufficiently smooth functions. The method consists of dividing the inner and outer region and solving the inner region problem by a fourth order cubic spline after rescaling the inner region. The solution of the reduced problem provides the solution of the outer region problem. ~~Since~~ the point of division of inner and outer region is arbitrary and also it is observed that the solution obtained is discontinuous at the point of division. In an attempt to overcome these drawbacks, a more general method namely Method 2 is derived, which is also applicable to Non-Linear Problems of the form:

$$\varepsilon y' = f(x,y,\varepsilon);\ y(a) = \alpha,\ a \le x \le b,$$

where $\frac{\partial f}{\partial y}(x,y,0)$ is non-zero on (a,b) and has only one stable root for all values of x and y. The method consists of a numerical cutting point technique. This method also takes care of continuity at the cut point. Both these methods provide global fourth order approximation to the exact solution and does not depend on asymptotic expansion.

In chapter III, A class of Non-Linear Two-Point Singularly Perturbed Boundary Value Problems is considered.

$$\varepsilon y''(x) + [p(y(x))]' + q\,(x,y(x)) = r(x);\ y(a) = \alpha,\ y(b) = \beta;$$

here, $0 < \varepsilon << 1$; α and β are given constants; $p(y)$, $q(x,y)$, $r(x)$ are assumed to be sufficiently differentiable functions. Furthermore, we assume that the problem has a solution which displays a boundary layer of width $o(\varepsilon)$ at $x = a$ for small value

of ε. The original boundary value problem is reduced to an asymptotically equivalent following initial value problem by an initial value technique:

$\varepsilon y'(x) + p(y(x)) + Q(x) = R(x) + K;\ y(a) = \alpha$, K a constant.

Then it is solved in the inner region by cubic spline method for inner region employed in Method 2 developed in chapter 2. The solution of the reduced problem:

$$[p(y_o(x))]' + q(x, y_o(x)) = r(x);\ y_o(b) = \beta,$$

provides the solution in the outer region.

In chapter IV, we first consider the following class of general Linear Singularly Perturbed Boundary Value Problems without first derivative term.

$$\varepsilon\, y'' = q(x)\, y + r(x);\ y(a) = \alpha_0,\ y(b) = \alpha_1,$$

where $q(x) > 0$ and $r(x)$ are sufficiently smooth functions.

Ahlberg's cubic spline in terms of moments is considered. A second order variable mesh finite difference scheme is developed using the continuity conditions of second derivatives of the spline at the interior nodes. Then the variable mesh scheme is generalised for the following class of problems with first derivative term:

$\varepsilon\, y'' = p(x)y' + q(x)\, y + r(x);\quad y(a) = \alpha_0,\ y(b) = \alpha_1,$

where $p(x)$, $q(x)$ and $r(x)$ are sufficiently smooth functions.

In deriving the difference scheme for above class of problems, we have taken some second order approximations of first derivative terms in terms of solution of the problem at the nodal points.

In Chapter V, The following class of general Semi-Linear Singularly Perturbed Boundary Value Problems is considered.

$\varepsilon y'' = f(x,y); \; y(a) = \alpha, \quad y(b) = \beta; \qquad a < x < b, \quad 0 < \varepsilon << 1.$

It is known that, under suitable assumptions, one expects the solution to behave qualitatively as the solution of the following problem:

$$\varepsilon y'' = p(x)y + q(x); \; y(a) = \alpha, \; y(b) = \beta; \; p(x) \geq p > 0,$$

where $p(x)$ and $q(x)$ are sufficiently smooth functions.

Therefore, first we consider the above linear singularly perturbed problem. A family of variable mesh methods based on cubic spline approximations which gives third order approximations to the solution of the linear problem. Then these methods are used for solving semi-linear problems using a quasi-linearisation technique. When a parameter namely mesh ratio parameter present in the methods is taken to be unity, the family of third order methods reduces to a family of fourth order methods, and in addition, if other parameter is also taken to be unity, the family of methods reduces to a well known Numerov's method for corresponding regular problem.

In Chapter VI, the variable mesh methods based on cubic spline approximations developed in the previous chapter are generalised for the general Non-linear Singularly Perturbed Boundary Value Problems:

$\varepsilon y'' = f(x,y,y'); \; y(a) = A, \; y(b) = B; \; x \in (a,b), \; 0 < \varepsilon \ll 1.$

CHAPTER I

INTRODUCTION

1.1 GENERAL:

Singular Perturbation, now is a maturing mathematical subject with a fairly long history and a strong promise for continued important applications throughout science and engineering. We give the brief definition of a singularly perturbation problem in its simplest and most widely used form. Consider a problem P_ε in some differential model, depending on a small positive parameter ε, where $0 < \varepsilon << 1$. Under some conditions, a solution $y_\varepsilon(x)$ of the problem P_ε can be constructed by the well known method of perturbation i.e. as a power series in ε with first term y_0 being the solution of the problem P_0, which is obtained by putting ε equal to zero in the problem p_ε. Under the happiest circumstances, this perturbation method leads to altogether satisfactory results. This series cannot often be presumed to uniformly converge, particularly for small values of ε, in the entire interval. when such an expansion converges as $\varepsilon \longrightarrow 0$, uniformly in x, one speaks of 'Regular Perturbation Problem'. On the other hand, when $y_\varepsilon(x)$ does not have a uniform limit in x as $\varepsilon \longrightarrow 0$, this straight forward perturbation method fails and as a consequence of the non-uniformity, one may miscalculate or even loss essential results, one then speaks of 'Singular Perturbation Problem'. A Singular Perturbation problem is well defined as one in which no single asymptotic expansion is uniformly valid throughout the

interval, as $\varepsilon \longrightarrow 0$. The prototype of Singular Perturbation Problem is Prandtl's boundary layer theory, which provides a beautiful mathematical tool for the investigation of a practical problem like Navier-stokes equations with large reynold number, one of the most striking example of singular perturbation. By definition, the boundary layer is a narrow region where the solution of a differential equation changes rapidly and the thickness of boundary layer must approach zero as ε tends to zero.

The singular perturbation problems find place in many area of engineering and applied mathematics, for instance: fluid mechanics, fluid dynamics, quantum mechanics, plasticity, chemical reactor theory, magneto hydrodynamics, reaction-diffusion processes and many other problems of fluid motion. A few notable examples are boundary layer and WKB problems. The problems to be solved by means of asymptotic and numerical analysis are becoming progressively more and more complicated.

The aim of this chapter is to present a brief survey of asymptotic and numerical analysis of Singular Perturbation Problems. A brief survey on the solution behavior of singular perturbation problems is also included. A summary of some recent methods is presented. Finally, the summary of the thesis is presented.

1.2 ASYMPTOTIC ANALYSIS OF SINGULAR PERTURBATION PROBLEMS:

In the first part of the century, although Birkhoff(1908), Langer(1931) and others have done work on asymptotic analysis of Linear Ordinary Differential Equations and significant amount of work on turning point problems being done by physicists

Wentzel(1926), Krammers(1926), Brillouin(1926) and others, but Friedrichs and Wasow seem to be the first mathematicians to study asymptotic solution of singularly perturbed boundary value problems. They first use the term 'Singular Perturbation' in print in the title of Friedrichs and Wasow(1946). Their work was motivated by an analysis of the edge effect for buckled plates. Soon afterwards, many authors like Tikhonov(1948), Levinson(1950), and Vasileva and Volosov(1967), began studying related problems. Levinson started study of wide range of important topics in Singular Perturbation and made contribution to the area together with number of young collaborators including Coddington, Davis, Flatto, Haber and Levin. The Russian school also did outstanding work on many subjects including boundary layer methods — Vishik and Lyusternik(1961). From around 1950, fluid dynamists solved some very interesting problems like linoleum-rolling problem — Carrier(1953) and low reynold number flow past bodies — Kaplaun(1957). At Caltech's Guggenheim Aeronautical Laboratory, Lagerstrom, Cole, Latta, Van Dyke, Kaplaun and others became equally involved in asymptotic expansion procedures for more general singular perturbation problems. An over-simplified matching procedure is presented in the book of Van Dyke(1964). The straight forward recipe he provided made it easy for tremendous variety of scientists to learn the rudiments of matching and to the important problem in their own disciplines. The basic idea, much as in Friedrichs lectures and Erdeyli(1956) lectures, involved an asymptotic matching of the inner and outer expansions at the edge of the boundary layer, where they should be appropriate. Cole(1968)

stressed the limit process expansions and two timings in a context far broader than fluid mechanics. Indeed the results obtained through matching generally coincided with those known through the intuitive folkways of the various fields. Wasow(1965) placed the singular perturbation in the contexts of the analytic theory of differential equations. By 1970, courses in perturbation methods became common in science, engineering and applied mathematics departments, and inevitably a string of text books and high level monograph began to appear. They include, Erdelyi(1956), Bellman and Cooke(1963), Bellman(1964), Van Dyke(1964), Kaplaun(1967), Carrier and Pearson(1968), Ames(1972),El'sgol'ts and Norkin(1973), Dingle(1973), Willoughby(1974), O'Malley(1974), Aziz(1975), Brauner et al. (1977), Driver(1977), Bender and Orszag(1978), Eckhaus(1973,1979), Na(1979), Childs et al.(1979), Hughes(1979), Hemker and Miller(1979), Axelsson et al.(1979), Meyer and Parter(1980), Doolan et al.(1980), Nayfey(1981), Kevorkian and Cole(1981), Miranker(1981), Eckhaus and de Jager(1982), Ardema(1983), Verhlust(1979,1983), Miller(1975,1976,1980,1982).

In O'Malley 's (1982) book review, an outline of history of asymptotic methods for singular perturbation problems is given.

1.3 SOLUTION BEHAVIOR OF SINGULAR PERTURBATION PROBLEMS:

Many authors have studied the behavior of singularly perturbed problems also.

Hoppensteadt(1966) has discussed the behavior of the singularly perturbed problems on infinite interval. They considered the initial value problems of the form

$$x' = f(t,x,y,\varepsilon); \quad x(0) = x_0$$
$$\varepsilon y' = g(t,x,y,\varepsilon); \quad y(0) = y_0 \tag{1.1}$$

where $0 < \varepsilon << 1$ and x, f are real K-dimensional vectors with components $x = (x_1, x_2, \ldots, x_k)$ and $f = (f_1, f_2, \ldots, f_k)$, respectively and y and g are real J-dimensional vectors with components $y = (y_1, y_2, \ldots, y_J)$ and $g = (g_1, g_2, \ldots, g_J)$, respectively. The purpose of this paper was to investigate the behavior of solutions of (1.1) as $\varepsilon \longrightarrow 0$, for $t_0 \le t \le \infty$.

Dorr and Parter (1970) have discussed the asymptotic behavior as $\varepsilon \longrightarrow 0$ of solutions $u(t) = u(t,\varepsilon)$ and $v(t) = v(t,\varepsilon)$ to nonlinear boundary value problems of the form

$$u'' = f(t,u,v) \ ; \quad 0 < t < 1, \quad u(0) = u(1) = 0 \tag{1.2}$$

$$\varepsilon v'' + g(t,u,u')v' - C(t,u,u') = 0$$
$$0 < t < 1, \ v(0) = v_0; \ v(1) = v_1 \tag{1.3}$$

where, they have assumed $0 \le v_0 \le v_1$ and $C(t,u,u') \ge 0$. They were particularly interested in problems in which there is exactly one interior turning point for the equation(1.3), that is, for each $\varepsilon > 0$ there is unique point $\alpha \in (0,1)$ such that $g(\alpha,u(\alpha),u'(\alpha))=0$ and $g(t,u(t),u'(t))$ changes sign in a neighbourhood of $t = \alpha$. In order to study the asymptotic behavior of the solutions they assumed the following conditions :

(i) $0 \le v_0 \le v_1$,

(ii) $C(t,u,u') \ge 0$

(iii) $f(t,u,u')$, $g(t,u,u')$ and $C(t,u,u')$ are continuous in all variables

(iv) There exists a continuous function $f_0(t,v)$ such that

$| f(t,u,v) | \le f_0(t,v)$ for $t \in [0,1]$ and $v \in [0,v_1]$.

Hoppensteadt(1971), investigated the properties of solution of ordinary differential equations with a small parameter. In this paper they included the case in which the boundary conditions also depend on ε i.e. $x(t_0) = \alpha(\varepsilon)$ and $y(t_0) = \beta(\varepsilon)$. They also extended these results to the boundary value problems.

Cohen(1973) has discussed the existence and asymptotic behavior for small $\varepsilon > 0$ of the solution of the nonlinear two-point boundary value problem

$$\varepsilon y'' + f(x,y,y')y' = 0, \qquad 0 < x < 1 \tag{1.4}$$

with

$$y'(0) - ay(0) = A \ge 0 \qquad (a > 0) \tag{1.5a}$$

$$y'(1) + by(1) = B > 0 \qquad (b > 0) \tag{1.5b}$$

They have imposed the following conditions on the nonlinear f in (1.4)

(i) $f(x,y,y')$ is continuously differential in the region

$R = \{ (x,y,y'): 0 \le x \le 1, 0 \le y \le B/b, y' \ge 0\}$

(ii) $y_1 \le y_2$ and $y_1' \le y_2'$ imply that

$f(x,y_1,y_1') \le f(x,y_2,y_2')$ on R

(iii) $f(x,y,y') \ge \beta > 0$ on R

(iv) There exists a constant k such that for all $(x,y,y') \in R$

$| f(x,y,y') - f(x,z,z') | \le k(|y - z| + | y'-z'|)$.

Under these conditions, they proved that there exists a solution $y(x,\varepsilon)$ of (1.4),(1.5a & 1.5b) such that $y(x,\varepsilon)$ tends to B/b and $y'(x,\varepsilon)$ tends to zero on any subinterval $0 < \delta \le x \le 1$. The entire

analysis is based on the so-called 'shooting method' for ordinary differential equations.

Kriess and Parter(1974) have discussed the behavior of the solutions of the boundary value problems with turning points:

$$\varepsilon y''(x) + f(x,\varepsilon)y'(x) + g(x,\varepsilon)y(x) = 0 \tag{1.6}$$

for $-a \le x \le b$ with $y(-a) = A$ and $y(b) = B$ (1.7)

where $a,b > 0$, $\varepsilon > 0$ and $f(x,\varepsilon)$ has a simple zero in $[-a,b]$.

Sannuti(1975) has considered a class of two-point boundary value problems

$$\begin{aligned} x' &= g_1(x,t) + B_1(t)z + C_1(t)u \\ \lambda y' &= g_2(x,t) + B_2(t)z + C_2(t)u \end{aligned} \tag{1.8}$$

with $x(0) = x_0$ and $z(0) = z_0$ (1.9)

where x and z are n and m dimensional state vectors, respectively, u is an r-dimensional control vector, and λ is a nonnegative scalar parameter and g_1, g_2, B_1, B_2, C_1, C_2 are assumed to be infinitely differentiable in all their arguments in an appropriately defined domain, which arise in fixed final time free end point optimal control problems.

Harris(1976) has described the applicability of differential inequalities to singular perturbation problems by studying the model nonlinear singular perturbation problem

$$\varepsilon y'' + yy' - y = 0; \quad 0 \le x \le 1 \tag{1.10}$$

$$y(0) = A \text{ and } y(1) = B \tag{1.11}$$

whose solution exhibit a wide variety of interesting behavior.

Kopell and Parter(1981) have discussed the analysis of the problem

$$\varepsilon y''(t,\varepsilon) = [y^2(t,\varepsilon) - t^2]y'(t,\varepsilon) \tag{1.12}$$

with $y(-1,\varepsilon) = A$, and $y(0,\varepsilon) = B$ (1.13)

based entirely on priori estimates and the 'shooting method'.

Grasman and Maktowsky(1977) have employed a variational formulation of the problem (1.12) with (1.13) to resolve the question of the number of location of the boundary layers as well as to uniquely determine the asymptotic expansion of the solution. These results are extended to analogous problems for partial differential equations with turning points.

Nipp(1983) has studied an extension of Tikhonov's theorem in singular perturbations. They considered the autonomous system of ordinary differential equations.

Finden(1983) has presented an asymptotic approximation for solving singular perturbation problems. The given system

$$\begin{aligned} x' &= f(t,x,y,\varepsilon) \\ \varepsilon y' &= g(t,x,y,\varepsilon) \end{aligned} \tag{1.14}$$

together with initial conditions

$$x(0) = \alpha,\ y(0) = \beta \tag{1.15}$$

where x, f and α are m-dimensional vectors and y, g, β are n-dimensional vectors, is replaced by the following system which is not stiff numerically,

$$\begin{aligned} x' &= f(t,x,y,\varepsilon) \\ y' &= h(t,x,y,\varepsilon) \qquad \text{for some function } h. \end{aligned} \tag{1.16}$$

They have proved that the solution to (1.16) is a second order approximation for the quantity $g(x,y,t)/\varepsilon$ is given by

$$\frac{g(x,y,t)}{\varepsilon} = h(x,y,t,\varepsilon) + O(\varepsilon^2). \tag{1.17}$$

Dadfar et al.(1984) has constructed and analysed a power series expansion in the damping parameter ε of the limit cycle $U(t,\varepsilon)$ of the free Van der Pol equation

$$U'' + \varepsilon(U^2 - 1)\, U' + U = 0 \tag{1.18}$$

Grasman(1979) has discussed similar results for a class of elliptic singular perturbation problems.

Howes(1979) has discussed the boundary layer behavior of singularly perturbed initial value problems and the third order boundary value problems in Howes(1982).

Howes(1984) presented some results and sufficient condition in order that the solution of the problem(scalar and vector form)

$$\varepsilon y'' = F(y)y' + g(x,y), \quad a \le x \le b \tag{1.19}$$

$$\text{with } y(a,\varepsilon) = \alpha \text{ and } y(b,\varepsilon) = \beta, \tag{1.20}$$

display boundary layer behavior at an end point.

Axelsson and Carey(1985) considered a class of singularly perturbed boundary value problem with various type of boundary conditions and examined the boundary layer behavior of the solution. By constructing a more regular modified problem with a correction term, they discussed how to deal with the boundary layer separately and gave error estimates which are uniform in singular perturbation parameter.

1.4 NUMERICAL ANALYSIS OF SINGULAR PERTURBATION PROBLEMS:

The asymptotic analysis treats restrictive class of problems and require the problem solver to have some understanding of the behavior of the solution expected. As far as an asymptotic analysis is valid and a few terms in the asymptotic expansion describe the solution sufficiently accurate, one can rely on standard techniques to obtain the solutions. As and when the asymptotic analysis is difficult to handle or perform badly, then one can usually look for an alternative approach and because of this numerical analysis for solving singular perturbation problems came into existence. We would say that numerical analysis tries to provide quantitative information about a particular problem, whereas asymptotic analysis tries to gain insight about the qualitative behavior of a family of problems and only semi quantitative information of any particular member of the family of problems. Numerical methods are intended for broad class of problems and are intended to minimize demand upon the problem solver. Pearson(1968a) was perhaps the first to attempt to give numerical solution of singular perturbation problems. After him, in later years, many efficient numerical methods were developed, which can be broadly classified as finite difference methods, finite element methods and spline approximation methods.

1.4.1 FINITE DIFFERENCE METHODS :

Pearson(1968a) was perhaps the first to attempt something like net adjustments in the difference schemes while treating singular perturbation problems. Basically, his idea was to use variable mesh

width. Based on the classical three-point difference scheme for a non-uniform mesh, Pearson gave the numerical solution of a great variety of singular perturbation problems of the form

$$L_\varepsilon[y] = \varepsilon\, y''(x) + a(x)\, y'(x) + b(x)y(x) = f(x), \qquad (1.21)$$

for $$-1 \le x \le 1 \text{ with } y(-1) = A \text{ and } y(1) = B \qquad (1.22)$$

where ε is a small parameter ($0 < \varepsilon \ll 1$). he used a simple variable mesh approximation

$$\frac{2\varepsilon\, [q_i y_{i+1} + p_i y_{i-1} - (p_i+q_i)y_i]}{p_i q_i (p_i+q_i)} + a(x_i)\, \frac{[q_i^2 y_{i+1} - p_i^2 y_{i-1} + (p_i^2 - q_i^2)y_i]}{p_i q_i (p_i+q_1)}$$

$$+ b(x_i)y_i = f(x_i) \qquad (1.23)$$

where $p_i = x_{i+1} - x_i$ and $q_i = x_i - x_{i-1}$.

This is a first order approximation, if we assume the mesh ratio q_i/p_i as constant. A difficult solution procedure is followed, which finally requires a few thousand mesh points in the interval [-1, 1]. The mesh must be properly chosen so that the solution of the difference equation approximates that of the differential equation. This is accomplished by iteratively adjusting the mesh spacing such that the mesh points are concentrated in the region where $y(x)$ changes rapidly. The procedure is first to solve the problem with a uniform mesh and a modest value of ε ($\varepsilon = 10^{-2}$ or 10^{-3}). Then insert new mesh points between adjacent points, say x_i and x_{i+1}, for which a certain predetermined tolerance is exceeded (e.g., $| y_{i+1} - y_i | > \delta$, δ prescribed). A smoothing process is carried out to avoid locally abrupt changes in the mesh intervals. Next the problem is solved by Gaussian elimination. Finally, the size of ε is decreased and the procedure is repeated, using the

mesh obtained from the preceding ε-step as the initial mesh. Obviously, this method is costly in terms of computer time even for simple linear problems.

Pearson [1968b] has also given the solution of the nonlinear problem

$$F(x,y,y',\varepsilon y'') = 0 \qquad \text{for } 0 \le x \le 1 \tag{1.24}$$

$$\text{with } y(0) = \alpha \text{ and } y(1) = \beta \tag{1.25}$$

The same approximation as in case of linear problems is used for y' and y''. The resulting non-linear algebraic equations were solved by the Newton-Raphson iterative scheme. For example, to solve the non-linear problem

$$\varepsilon\, y'' + y'^2 = 1 \tag{1.26}$$

$$y(0) = y(1) = 1 \tag{1.27}$$

where $\varepsilon = 10^{-6}$, Pearson used 4000 mesh points to get a solution which agreed up to five significant digits with the exact solution. In a more difficult problem, he used 25000 points to get an accurate solution. In order to overcome the numerical instability of the standard methods the upstream (upwind) one-sided (directional) differences on a uniform mesh is introduced. For detailed discussion see Dorr(1970). The essential of this type of scheme is to replace the first order derivatives in (1.21) by a one-sided difference quotient instead of the central difference. The choice of a backward or forward difference depends on the sign of $a(x)$ at the particular mesh point under consideration, that is

$$\delta[ay_i] = \begin{cases} a(x_i)(y_{i+1} - y_i)/h & \text{if } a\,(x_i) \ge 0 \\ a(x_i)(y_i - y_{i-1})/h & \text{if } a\,(x_i) < 0 \end{cases} \tag{1.28}$$

Then the difference scheme takes the form

$$L_h[Y_i] = \varepsilon \left[\frac{y_{i+1}-2y_i+y_{i-1}}{h^2}\right] + \delta\,[\,ay_i] + b(x_i)y_i = f_i \qquad (1.29)$$

If $b(x_i) \le 0$ then it can been shown that L_h is a difference operator of the positive type and hence there exits a unique solution for each set of given data and for each $\varepsilon > 0$, $h > 0$. By a difference operator of positive type we mean a operator L_h of the form

$$L_h[Y_i] = A_i y_{i-1} + B_i y_i + C_i y_{i+1} \qquad (1.30)$$

where

(i) $A_1 > 0$, $\forall$ i, and (ii) $A_i + B_i + C_i \geqslant 0$, $\forall$ i.

With these above restrictions on $b(x)$, L_h satisfies a discrete maximum principle. Moreover, if the directional difference method is used, then for fixed $h > 0$ there is a limit function, as ε tends to zero, which satisfies the 'reduced' difference equation, (1.29) with $\varepsilon = 0$. One drawback of this method is that it is only of first order.

Dorr et al.(1973) have developed the idea further and they discussed the applications of the maximum principle to obtain elementary estimates for solution of second order ordinary differential equations. These estimates are applied to obtain results on the limiting behavior of solutions of singularly perturbed problems. They considered the linear problems under various hypothesis, including turning point problems, and qausilinear problems of the form:

$$\varepsilon y'' + \alpha(t,y(t),\varepsilon)y'(t) = \beta(t,y(t),\varepsilon) \tag{1.31}$$

for $a < t < b$ with $y(a) = A(\varepsilon)$ and $y(b) = B(\varepsilon)$ (1.32)

Il'in (1969) constructed a finite difference scheme which represent the rate of decay in the boundary layer for the equation of the form:

$$\varepsilon y'' + a(x)y' = f(x), \tag{1.33}$$

where $a(x) \equiv a$ is constant,

and then it is generalised for slightly more general problem i.e., when $a(x)$ is a variable and for a uniform mesh, the following difference operator is obtained,

$$L_h[y_i] = \gamma_i \left[\frac{y_{i+1} - 2y_i + y_{i+1}}{h^2} \right] + a(x_i) \left[\frac{y_{i+1} - y_{i-1}}{2h} \right] = f(x_i) \tag{1.34}$$

where γ_i is chosen so that the scheme is $O(h^2)$ accurate and correctly represent the rate of decay from the boundary layer into the interior. He found that for the eq(1.33),

$$\gamma_i = \frac{a(x_i)h}{2} \coth \left[\frac{a(x_i)h}{2\varepsilon} \right] \tag{1.35}$$

and so the operator $L_h[y_i]$ is of positive type.

He then proved the if a and f are C^2 on the interval, then

$$| y(x_i) - y_i | \le K h^2, \tag{1.36}$$

where K is a constant which depends on ε.

Abrahamsson et. al.(1974) have also given finite difference scheme for singular perturbation problems where the equation is a system of the form:

$$\varepsilon y'' + A(x)y' + B(x)y = F(x) \qquad 0 \le x \le 1 \tag{1.37}$$

with boundary conditions

$$y(0) = \alpha \text{ and } y(1) = \beta \tag{1.38}$$

where $\varepsilon > 0$ is a small parameter, $y = (y^{(1)}, y^{(2)}, \ldots y^{(n)})^T$ and $F = (F^{(1)}, F^{(2)}, \ldots F^{(n)})^T$ are vector functions with n components and $A, B \in C^\infty$ are n-rowed square matrices. They assumed that

$$A(x) = \begin{pmatrix} A^I(x) & 0 \\ 0 & A^{II}(x) \end{pmatrix}$$

with $A = A^*$(=adjoint of A), $A^I \le -\eta < 0$, $A^{II} \ge \eta > 0$ for $0 \le x \le 1$

They devised some schemes which with net spacing $h >> \varepsilon$, yield 'accurate' solutions in the interior, that is away from boundary layers. This is done and Richardson Extrapolation is justified under appropriate assumption on $A(x)$ but in general the accuracy cannot be better than $O(\varepsilon)$. In the scalar case, their work can be considered as refinement of upstream one-sided difference scheme. The idea is to introduce a parameter into the difference equation in such a way that more accurate approximation for the reduced problem is obtained.

Nijima(1978) derived a difference scheme for a semilinear problem with any number of turning points of arbitrary orders. The class of the problems considered are

$$\varepsilon y'' - [a(x)y]' - b(x,y) = 0; \qquad 0 \le x \le 1 \tag{1.39}$$

$$\text{with } y(0) = A \text{ and } y(1) = B \tag{1.40}$$

They showed that a solution of their scheme converges uniformly in ε, to that of continuous problem.

Lorenz(1979) has considered the numerical solution of singularly perturbed two-point boundary value problem of the form :

$$-\varepsilon y'' + \alpha(x,y)y' + \beta(x,y) = 0 \tag{1.41}$$

for $0 \le x \le 1$ with $y(0) = A$ and $y(1) = B$ (1.42)

He essentially breaks the interval into two parts and uses different meshes on each part to solve a reduced and boundary layer problem. It is noted that the criteria for choosing the break point is not given although the cutting parameter cannot be too large. Further the solution of the reduced problem is used as the data for the boundary layer problem at the cut point. Thus Lorenz solves the reduced problem only upto the break point and then solves the boundary layer problem for the rest of the interval.

Kriess and Kriess(1981) have given some finite difference schemes for a system of the form :

$$\frac{dy}{dx} = A(x)y + F(x) \qquad 0 \le x \le 1 \tag{1.43}$$

with n linearly independent boundary conditions

$$R_0 y(0) + R_1 y(1) = g \tag{1.44}$$

where $y = (y^{(1)}, y^{(2)}, \ldots y^{(n)})^T$ is a vector function with n components and R_0, R_1 and $A(x)$ are n - rowed square matrices.

Osher(1981) has developed upwind finite difference approximation for non-linear problems and also for system of non-linear hyperbolic conservation laws. Similar discussions with one sided difference schemes for the problem

$$\varepsilon y'' - a(y)' - b(x,y) = F(x); \quad -1 \le x \le 1 \tag{1.45}$$

with $\qquad y(-1) = A$ and $y(1) = B$ (1.46)

where A,B are constants, $a(y)$,$b(x,y)$ are C^2 functions with $b(x,0) = 0$ and $\frac{\partial b(x,y)}{\partial y} \geq 0$ is contained in Osher(1981).

Abrahamsson and Osher(1982) have discussed the monotone difference schemes for the problem

$$\varepsilon y'' - [f(y)]' - b(x,y) = 0, \qquad 0 \leq x \leq 1 \tag{1.47}$$

with

$$y(0) = A \text{ and } y(1) = B \tag{1.48}$$

where $f \in C^1(\mathbb{R})$ and $b \in C^1(\mathbb{R}^2)$ and $\frac{\partial b(x,y)}{\partial y} \geq \delta > 0$.

They proved some convergence results and showed that the Engquist-Osher(1981) monotone scheme will reproduce essential properties of the true solution for any grid.

Hoppensteadt and Miranker(1983) have discussed an extrapolation method for the numerical solution of singular perturbation initial value problems of the form:

$$\begin{aligned} \frac{dx}{dt} &= f(t,x,y,\varepsilon); \quad x(0) = \xi(\varepsilon) \\ \varepsilon\,\frac{dy}{dt} &= g(t,x,y,\varepsilon)\ ; \quad y(0) = \eta(\varepsilon) \end{aligned} \tag{1.49}$$

Sakai(1975) and Sakai and Usmani(1984) have discussed some numerical method based on chopping procedures for the solution of singularly perturbed two-point boundary value problems.

Niijima(1984) has derived a completely exponentially fitted difference scheme for turning point problems.

O'Malley and Flaherty(1980) have discussed some analytical and numerical methods for nonlinear singular singularly perturbed initial value problems. These results are extension of Ascher and Weiss(1983,1984a,1984b). Singular-singularly perturbed boundary value problems are also considered by O'Malley (1979).

Roberts(1982) has given boundary value technique to solve singular perturbation problems. It is based on the shooting methods.
Roberts(1983) has also discussed the analytic and approximate solutions of the problem:

$$\varepsilon y'' = yy', \quad -1 \le x \le 1 \tag{1.50}$$

$$y(-1) = \alpha \text{ and } y(1) = \beta \tag{1.51}$$

Roberts(1984) has extended his boundary value technique to solve the problem:

$$\varepsilon y'' + yy' - y = 0; \quad 0 \le x \le 1 \tag{1.52}$$

$$y(0) = \alpha \text{ and } y(1) = \beta \tag{1.53}$$

Jain et al.(1984) have given a family of third order variable mesh difference methods for two-point, second order, singular perturbation problems of the form:

$$y'' = f(x,y,y') \tag{1.54}$$

where f contains the parameter implicitly,Under the natural, mixed and nonlinear boundary conditions. when the mesh ratio is equal to unity, then these third order variable mesh method reduces to a unique fourth order method.
Vulanovic(1989) has considered finite difference scheme for the following singularly perturbed boundary value problem:

$$Tu \equiv -\varepsilon y'' + b(u)u' + c(x,u) = 0, \quad x \in I = [0,1] \tag{1.55}$$

$$Bu \equiv (u(0),u(1)) = (U_0,U_1) \tag{1.56}$$

Under the assumptions;

H1. Let k = 1 or K = 2 and let

$b \in C^k(W)$, $c \in C^k(I \times W)$, where $W = [u_*,u^*]$;

and $u_* < u^*$ satisfy:

H2. $C(x,u^*) \geq 0 \geq C(x,u_*)$, $x \in I$

$u^* \geq U_j \geq u_*$, $j = 0,1$

and $C_u(x,u) \geq C_* > 0$, $x \in I, u \in W$

Vulanovic(1989) has also given a numerical method for quasilinear singular perturbation problems of the following form:

$$Tu \equiv -\varepsilon u'' + b(x,u)u' + c(x,u) = 0, \qquad x \in I = [0,1] \tag{1.57}$$

$$Bu \equiv (u(0),u(1)) = (U_0,U_1) \tag{1.58}$$

under the assumptions,

$$b,c \in C^2(I\times\mathbb{R}),$$

$$b(x,u) \geq \beta > 0,\ b_x(x,u) + c_u(x,u) \geq \gamma,\ \gamma \leq 0,\ x \in I,\ u \in \mathbb{R},$$

$$\beta^2 + 4\varepsilon\gamma > 0$$

The continuous problem is transformed to the form:

$$\varepsilon u'' + f(x,u)' = g(x,u), \tag{1.59}$$

where

$$f(x,u) = \int b(x,s)ds,\ g(x,u) = f_x(x,u) + c(x,u)$$

Then new variables (t,y), are introduced which are given by $x = \lambda(t)$, $y = u(\lambda(t))$, where $\lambda \in C^2(I)$, $\lambda'(t) > 0$ for $t \in I$ and $\lambda(0) = 0$, $\lambda(1) = 1$. As a result of transformation following equation is obtained:

$$Py \equiv -\varepsilon(\mu(t)y')' - p(t,y)' + q(t,y) = 0, \tag{1.60}$$

$$By = (U_0,U_1) \tag{1.61}$$

where $\mu(t) = 1/\lambda'(t)$, $p(t,y) = f(\lambda(t),y)$, $q(t,y) = g(\lambda(t),y)\lambda'(t)$.

Then a finite difference scheme is given and first order convergence in the perturbation parameter is proved in discrete L^1-norm.

Herceg(1990) has developed a uniform fourth order difference scheme for the singularly perturbed problem of the form:

$$T_\varepsilon \equiv -\varepsilon^2 u'' + c(x,u) = 0, \quad x \in \ell = [0,1], \tag{1.62}$$

$$u(0) = u(1) = 0 \tag{1.63}$$

under the following assumptions:

$$c \in C^k(\ell \times \mathbb{R}),\ k \in N,$$

$$g(x) \le c_u(x,u) \le G(x), \qquad (x,u) \in \ell \times \mathbb{R},$$

$$\delta = \min \{ 5g(x) - 2G(x) : x \in \ell \} > 0,$$

$$0 < \gamma^2 \le g(x), \quad | g'(x) | \le L, \quad | G'(x) | \le L, \quad x \in \ell.$$

1.4.2 <u>FINITE ELEMENT METHODS</u> :

Zienkiewicz et al. (1975) were the first to treat singular perturbation problems by finite element method. They have stated that a finite element equivalent to upwind difference was needed to avoid the problems of oscillatory solutions obtained with standard numerical approximations. A further algorithm using piecewise linear triangular elements in two dimensions was proposed by Tabata(1977) and a further evaluation of the methods as well as a comparison with equivalent finite difference formulations for the convective-diffusion equation has been given in Zienkiewicz and Heinrich(1978). Some application of theory of Babuska(1973,1981) to the singular perturbation problems have been published by Reinhardt(1982).

Major contribution to the application of finite element methods for singular perturbation problems have come from Mitchell, Zeinkiewicz, Babuska, Hemker, Miller, Griffiths, Van Veldhuizen, Reinhardt and Christie.

Miller(1975) has given the concept of introducing a parameter into the numerical scheme and then choosing it in order to meet some criterion for singular perturbation problems of the form:

$$-\varepsilon u'' + a_1 u' + a_0 u = f \text{ in } \Omega = (0,1) \tag{1.64}$$

$$\text{with } u(0) = u(1) = 0 \tag{1.65}$$

where $a_0 \geq 0$ and $a_1 \geq 0$ are constants. He treats this problem by means of finite element method. The parameter $\theta = \theta(\varepsilon)$ is chosen so that

$$0 \leq \theta \leq 1, \ \lim_{\varepsilon \to 0} \theta = 0, \ \lim_{\varepsilon \to 0} \frac{a_1 \theta h}{2\varepsilon} = 1 \tag{1.66}$$

where h is the uniform mesh size.

This parameter θ is introduced in the basis $\{\phi_j\}_1^{N-1}$ as follows:

$$\phi_j(x) = \begin{cases} (x-x_{j-1})/\theta h & \text{for } x \in [x_{j-1}, x_{j-1}+\theta h] \\ 1 & \text{for } x \in [x_{j-1}+\theta h, x_j] \\ (x_j+\theta h-x)/\theta h & \text{for } x \in [x_j, x_j+\theta h] \\ 0 & \text{otherwise} \end{cases}$$

It is then shown that the finite element scheme is equivalent to

$$\left[-\frac{\varepsilon}{\theta h} + \frac{a_1}{2}\right] u_{j+1} + \left[\frac{2\varepsilon}{\theta h} + a_0 h\right] u_j + \left[-\frac{\varepsilon}{\theta h} - \frac{a_1}{2}\right] u_{j-1}$$

$$= h\,[f(x_{j-1}+\theta h) + f(x_j)]/2, \ j = 1(1)N-1 \tag{1.67}$$

with $$u_0 = u_N = 0 \tag{1.68}$$

In the limit when $\varepsilon = 0$ and thus $\theta = 0$, because of (1.66), (1.67) becomes the upwind finite difference scheme for the reduced problem of (1.64),

$$a_1 \frac{u_j - u_{j-1}}{h} + a_0 u_j = \frac{f(x_{j-1}) + f(x_j)}{2}, \qquad j=1(1)N-1 \tag{1.69}$$

with $\qquad u_0 = 0 \qquad (1.70)$

In the limit when $\theta = 1$, (1.67) is the usual central difference for (1.64)

$$-\varepsilon \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} + a_1 \frac{u_{j+1} - u_{j-1}}{2h} + a_0 u_j = f(x_j) \tag{1.71}$$

A particularly interesting intermediate choice of θ is

$$\theta = \left[\tanh \frac{a_1 h}{2\varepsilon} \right] / \left[a_1 h/2\varepsilon \right] \tag{1.72}$$

Now, this θ satisfies (1.66) and (1.67) becomes

$$- \frac{a_1 h}{2} \coth\left(\frac{a_1 h}{2\varepsilon} \right) \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} + a_1 + \frac{u_{j+1} - u_{j-1}}{2h} + a_0 u_j$$

$$= \frac{f(x_{j-1} + \theta h) + f(x_j)}{2} \qquad j=1(1)N-1 \tag{1.73}$$

with $\qquad u_0 = u_N = 0 \qquad (1.74)$

which is the finite difference scheme introduced by Il'in(1969) Miller has not carried out an error analysis in this paper, but has extended the above concept for partial differential equations in Miller(1976).

Christie et al.(1976) has considered the following problem:

$$\varepsilon y'' - Ky' = 0, \qquad 0 < x < 1 \tag{1.75}$$

$$\text{with } u(0) = 1 \text{ and } u(1) = 0 \tag{1.76}$$

He has placed a parameter in the test functions instead of using the parameter in the elements for obtaining asymmetric linear and quadratic basis functions for the test space. In this way the

oscillations which occur with symmetric test functions are no longer present.

Veldhuizen(1978) has used a finite element method using piecewise polynomials of degree $\le k$ to approximate the problem

$$\varepsilon u'' + u' = f; \qquad 0 \le x \le 1 \tag{1.77}$$

$$\text{with } u(0) = u(1) = 0 \tag{1.78}$$

with a very irregular mesh. on this mesh, error estimates of order $O(h^{k+1})$ are obtained uniformly in ε, h the maximum step size, for $k \ge 2$. They have extended these results and discussed some higher order schemes of positive type for the following class of problems

$$\varepsilon u'' + a(x)u' = f(x) \ ; \quad 0 \le x \le 1 \tag{1.79}$$

with

$$u(0) = u(1) = 0 \tag{1.80}$$

where $a(x) \ge a_0 > 0$.

They have also extended these results for elliptic singularly perturbed problems in two dimensions.

Hsiao and Jordan(1979) have discussed numerical schemes based on the method of matched asymptotic expansion and modifying the boundary layer problem. The particular method they used for solving the modified problems in Galerkin method with linear finite elements as trial functions. They have applied these schemes successfully to several examples. Finally, an error analysis has been performed and encompasses both numerical and singular perturbation theory.

O'Riordan(1984) has introduced a new piecewise linear finite element, which is designed to handle singular perturbation problems. They first introduced the new concept of hinged elements. Then they examined finite element approximations, to the problems of the form

$$\varepsilon y'' + ay' = f; \qquad 0 \le x \le 1 \tag{1.81}$$

with

$$y(0) = y(1) = 0 \tag{1.82}$$

whose errors are uniformly in ε i.e. the error constants and asymptotic order of convergence as $h \longrightarrow 0$ are independent of ε. Piecewise exponential elements yield

$$\max_{0 \le i \le N} | y(x_i) - y^h(x_i) | \le Ch^2 \tag{1.83}$$

$$\| y-y^h \|_\infty \le Ch \tag{1.84}$$

where C is a constant independent of ε and h.

1.4.3 SPLINE APPROXIMATION METHODS :

Starting from late sixties, many researchers have used splines for solving regular perturbation problems and some have also used it for solving singular perturbation problems. In this section, we give a brief survey of these methods.

Loscalzo and Talbot(1967) were perhaps first to use spline for solving differential equations. They have given a procedure for obtaining spline function approximations for solutions of the initial value problems. He has used quadratic and cubic splines and showed that the method is related to well known trapezoidal rule and Milne-Simpson method and the method is shown to be divergent, when higher degree spline functions are used.

Bickley(1968) has used cubic splines for solving two-point boundary value problems. He has considered the general linear two point boundary value problems of the form:

$$p(x)u'' + q(x)u' + r(x) = s(x) \tag{1.85}$$

with boundary conditions

$$\alpha_0 u + \beta_0 u' = \gamma_0 \quad \text{at} \quad x = x_0 \tag{1.86}$$

$$\alpha_n u - \beta_n u' = \gamma_n \quad \text{at} \quad x = x_n \tag{1.87}$$

The essential feature of his analysis is that it leads to a set of linear equations whose matrix of coefficients is of upper Hessenberg form i.e. an upper triangular matrix with a single subdiagonal.

Soon afterward, Albasiny and Hoskin(1969) has used Ahlberg's cubic spline for obtaining the solution of the second order linear differential equation

$$y'' + f(x)y' + g(x)y = r(x) \tag{1.88}$$

with $$y(x_0) = a \text{ and } y(x_N) = b \tag{1.89}$$

He has considered separately the cases where first derivative term is absent and where the first derivative is present.

For the first case, using the continuity conditions of first derivatives at the nodal points, he has obtained the following tridiagonal scheme,

$$y_{j+1}\left(1 + \frac{h^2}{6} g_{j+1} \right) - y_j\left(2 - \frac{2h^2}{3} g_j \right) + y_{j-1}\left(1 + \frac{h^2}{6} g_{j+1} \right)$$

$$= \frac{h^2}{6}\left(r_{j+1} + 4r_j + r_{j-1} \right) \tag{1.90}$$

and for the second case, he obtained the following scheme

$$y_{j+1}\left(1 - \frac{h}{2} f_{j+1} + \frac{h^2}{6} g_{j+1} \right)A_j$$

$$- y_j\left[\left(1 + \frac{h}{2} f_{j+1}\right)A_j + \left(1 - \frac{h}{2} f_{j+1}\right)B_j - \frac{2h^2}{3} g_j C_j\right] + y_{j-1}\left(1 + \frac{h^2}{6} g_{j+1} \right)$$

$$+ y_{j-1}\left(1 - \frac{h}{2} f_{j-1} + \frac{h^2}{6} g_{j-1} \right)B_j = \frac{h^2}{6}\left(A_j r_{j+1} + 4C_j r_j + B_j r_{j-1} \right)$$

$$j = 1,2,\ldots,n-1 \tag{1.91}$$

where $A_j = \left(1 - \frac{h}{3} f_{j-1} \right) \left(1 + \frac{h}{3} f_j \right) + \frac{h^2}{36} f_{j-1} f_j$

$B_j = A_{j+1}$

$C_j = 1 + \frac{7h}{24} \left(f_{j+1} - f_{j-1} \right) - \frac{h^2}{12} f_{j-1} f_{j+1}$

Fyfe(1969) has examined the Bickley's method and obtained the error estimates. He also examined deferred corrections, effect of nonuniform spacing and a mesh refinement process.

Jain, P. C. and Holla(1978) has given a scheme for linear hyperbolic equations with variable coefficients, where they have approximated the time derivatives by central differences and the space derivatives by second order derivatives of interpolating cubic spline.

Patrico(1978) has presented a numerical process which provides $O(h^4)$ cubic spline approximate function approximation for the solution of general initial value problem.

Patrico(1979) has presented a numerical process which provides $O(h^6)$ spline approximate function approximation for the solution of general initial value problem. He has used spline functions of degree five for obtaining the approximate solution. He has shown

that the method is stable and convergence analysis is given.

Mastro and Voss(1981) have considered a quintic spline collocation procedure for solving the Falkner-Skan boundary layer equation.

Jain and Aziz(1981) have constructed the following parametric spline

$$S''(t) = -\frac{h^2}{w^2 \sin w}\Big[S''(t_\nu)\sin w\Big((t-t_{\nu-1})/h\Big) + S''(t_{\nu-1})\sin w\Big((t_\nu - t)/h\Big)\Big]$$

$$+\frac{h^2}{w^2}\Big[\Big((t-t_{\nu-1})/h\Big)\Big(S''(t_\nu) + \frac{w^2}{h^2}S(t_\nu)\Big)$$

$$+\Big((t_\nu - t)/h\Big)\Big(S''(t_{\nu-1}) + \frac{w^2}{h^2}S(t_{\nu-1})\Big)\Big], \tag{1.92}$$

where $w = h\sqrt{p}$, $p > 0$ and which reduces to cubic spline for $p = 0$. They have used this spline for solving certain ordinary differential equations and partial differential equations. They have compared results using parametric and cubic splines.

Chawla and Subramanian(1987) have given a fourth order method based on Numerov's method and cubic spline for the following problem:

$$y'' + f(x,y) = 0, \qquad 0 \le x \le 1, \tag{1.93}$$

$$y(0) = \alpha, \; y(1) = \beta \tag{1.94}$$

where $f(x,y)$ is continuous, $\partial f/\partial y$ exists and is continuous

First,They have extended Bickley's idea to the above non-linear problem and carried out analysis which shows that it is only second order accurate. Then they have given a fourth order method where Numerov's method is used to compute solution at the nodal points. These values of the solution are then used in continuity condition to obtain approximation of second order derivatives needed for construction of fourth order approximate cubic spline solutions.

They have presented numerical examples which demonstrate the fourth order accuracy of their method.

Chawla(1988) has extended the results of above paper for general two-point boundary value problem of the form:

$$y'' = f(x,y,y'); \; a < x < b \tag{1.95}$$

subject to boundary conditions

$$\alpha_0 y(a) - \alpha_1 y'(a) = A, \; \beta_0 y(b) + \beta_1 y'(b) = B, \tag{1.96}$$

$\alpha_0, \alpha_1, \beta_1$ and β_2 are suitable constants.

Surla and Stajonovic(1981) have give a difference scheme for singularly perturbed boundary value of the form:

$$-\varepsilon\, y'' + p(x)y = f(x), \; p(x) > 0, \tag{1.97}$$

$$y(0) = \alpha_0, \; y(1) = \alpha_1 \tag{1.98}$$

They have used the following spline in tension.

$$S_j(x) = u_j t + (1-t)u_{j-1} + M_j\,(h^2/p^2)\left(\frac{\sinh pt}{\sinh p} - t\right)$$

$$+ M_{j-1}\,(h^2/p^2)\left(\frac{\sinh p(1-t)}{\sinh p} - (1-t)\right), \tag{1.99}$$

$t = (x-x_{j-1})/h$, $x \in [x_{j-1}, x_j]$, $x_j = jh$, $j = 0(1)n$, $h = 1/n$,

$p = \bar{p}\, h$, $\bar{p} = (p/\varepsilon)^{1/2}$

They have obtained error of the form $O(h, \min(h, h\sqrt{\varepsilon})$ for the solution at the grid points and second order convergence for the solution at off-nodal points for the case $p(x) = p > 0$.

Sakai and Usmani(1989) have considered an application of simple exponential spline to the numerical solution of the following singular perturbation problem:

$$\varepsilon y'' + b(x)y' - d(x)y = f(x) \qquad 0 \le x \le 1 \tag{1.100}$$

$$y(0) = \alpha_0, \; y(1) = \beta \tag{1.101}$$

for small $\varepsilon > 0$ and for smooth data function b,d and f subject to the conditions $d(x) \ge 0$ and $b(x) \ge B > 0$ on $[0,1]$.

Stajonovic(1990) has considered the self-adjoint singular perturbation problem

$$Lu \equiv -\varepsilon u''(x) + p(x)u(x) = f(x), \; p(x) \ge p > 0, \; x \in (0,1) \tag{1.102}$$

$$u(0) = \alpha_0, \; u(1) = \alpha_1, \qquad \alpha_0, \alpha_1 \in \mathbb{R} \tag{1.103}$$

and has used the following exponential spline difference scheme for obtaining numerical solution of the above problem:

$$r_i^- v_{i-1} + r_i^c v_i + r_i^+ v_{i+1} = q_i^- f_{i-1} + q_i^c f_i + q_i^+ f_{i+1}$$

$$i=1(1)n-1 \tag{1.104}$$

$$v_0 = \alpha_0, \qquad v_N = \alpha_1 \tag{1.105}$$

where,

$$r_i^- = \lambda_{i-1}/2\sinh(\lambda_{i-1}/2), \quad r_i^+ = \lambda_{i+1}/2\sinh(\lambda_{i+1}/2),$$

$$r_i^c = -\lambda_i \cosh\lambda_i/\sinh(\lambda_i/2)$$

$$q_i^- = p_{i-1}^{-1}(1 - \lambda_{i-1}/2\sinh(\lambda_{i-1}/2)), \quad q_i^+ = p_{i+1}^{-1}(1 - \lambda_{i+1}/2\sinh(\lambda_{i+1}/2)),$$

$$q_i^c = p_i^{-1}(-2 + \lambda_i \cosh \lambda_i/\sinh(\lambda_i/2))$$

$$p_i = p(x_i), \; \lambda_i = (p_i/\varepsilon)^{1/2}h.$$

He has proved that for $p(x), f(x) \in C^2(0,1]$ and $p'(0) = p'(1) = 0$

$$| \, u(x_i) - v_i \, | \le M \min(h^2, \varepsilon) \tag{1.106}$$

where v_i denote the exponential spline solution and M is a constant.

Roos(1990) has considered a self-adjoint model singular perturbation problem

$$L_\varepsilon u \equiv -\varepsilon^2 u'' + p(x)u = f(x), \tag{1.107}$$

$$u(0) = u(1) = 0. \tag{1.108}$$

where it is assumed that $p(x) \geq p_0 > 0$ and p,f are continuous.
He has constructed global uniformly first and second order schemes using patched base spline functions by replacing the variable coefficients of the differential equation by piecewise polynomials and to solve the resulting problem exactly.

Bhatta and Sastri (1991) have given a seventh order spline procedure, where an octic spline coupled with a heptic spline function is used for solving the two-point boundary value of the form:

$$y'' = Py + Q \tag{1.109}$$

$$y(0) = \alpha, \; y(1) = \beta \tag{1.110}$$

where P and Q are continuous with $P > 0$, $x \in [0,1]$.

1.5 DEFINITIONS AND ALGORITHMS :

1.5.1 Definition : A tri~~angular~~diagonal matrix $M = (m_{ij})$ of order n is said to be irreducible if

$$m_{i,i-1} \neq 0 \qquad 2 \leq i \leq n$$

and

$$m_{i,i+1} \neq 0 \qquad 1 \leq i \leq n-1$$

1.5.2 Definition : A matrix $M = (m_{ij})$ of order n is said to be diagonally dominant if

$$\sum_{\substack{j=1 \\ j\neq i}}^{n} |\, m_{ij} \,| \leq |\, m_{i.i} \,|, \qquad i \leq i \leq n \tag{1.111}$$

and strictly diagonally dominant if strict inequality holds for all i

1.5.3 Definition : A matrix $M = (m_{ij})$ of order n is said to be <u>irreducibly diagonally dominant</u> if M is irreducible, diagonal dominant and strict inequality holds in (1.111) atleast for one i.

1.5.4 Definition : A matrix $M = (m_{ij})$ of order n is said to be <u>monotone</u> if $M^{-1} \geq 0$.

1.5.5 Definition : If a matrix $M = (m_{ij})$ is irreducibly diagonally dominant and has nonpositive off-diagonal elements, then M is monotone.

For solving the tridiagonal systems in this thesis, we have used the <u>Thomas Algorithm</u> which is described below :

1.5.6 <u>THOMAS ALGORITHM</u> : Consider the following tridiagonal system of order n

$$
\begin{array}{llllll}
d_1x_1 + c_1x_2 & & & & & = b_1 \\
a_2x_1 + d_2x_2 + c_2x_3 & & & & & = b_2 \\
\quad a_3x_2 + d_3x_3 + c_3x_4 & & & & & = b_3 \\
\quad \quad \cdots\cdots & & & & & \\
& a_{n-1}x_{n-2} + d_{n-1}x_{n-1} + c_{n-1}x_n & & & & = b_{n-1} \\
& \qquad\qquad a_nx_{n-1} + c_nx_n & & & & = b_n
\end{array}
$$

Assuming $d_1 \neq 0$, we eliminate x_1 from the second equation, getting the equation

$$d_2'x_2 + c_2x_3 = b_2'$$

with $d_2' = d_2 - \dfrac{a_2}{d_1} c_1, \quad b_2' = b_2 - \dfrac{a_2}{d_1} b_1,$

Next, assuming $d_2' \neq 0$, we use this equation to eliminate x_2 from the third equation, getting the new equation

$$d_3' x_3 + c_3 x_4 = b_3'$$

with $d_3' = d_3 - \dfrac{a_3}{d_2'} c_2$, $\quad b_3' = b_3 - \dfrac{a_3}{d_2'} b_2'$,

Continuing in this manner, we eliminate at the step k, x_k from the equation k+1(assuming that $d_k' \neq 0$), getting the new equation

$$d_{k+1}' x_{k+1} + c_{k+1} x_{k+2} = b_{k+1}'$$

with

$$d_{k+1}' = d_{k+1} - \frac{a_{k+1}}{d_k'} c_k$$

$$b_{k+1}' = b_{k+1} - \frac{a_{k+1}}{d_k'} b_k' \qquad \text{for } k = 1,2,\ldots,n-1$$

During back substitution, we first get, assuming $d_n' \neq 0$,

$$x_n = \frac{b_n'}{d_n'}$$

and then, for $k = n-1,\ldots,2,1$

$$x_k = \frac{b_k' - c_k x_{k+1}}{d_k'}$$

1.6 SUMMARY OF THE THESIS:

In chapter II, we first, present a numerical method namely Method 1, for Linear Singularly Perturbed Initial Value Problems. The method consists of dividing the inner and outer region and solving the inner region problem by a fourth order cubic spline after rescaling the inner region. The solution of the reduced problem provides the solution of the outer region problem. ~~Since~~ The point of division of inner and outer region is arbitrary and also it is

observed that the solution obtained is discontinuous at the point of division. In an attempt to overcome these drawbacks, a more general method namely Method 2 is derived, which is applicable to Non-Linear Problems also. The method consists of a numerical cutting point technique. This method also takes care of continuity at the cut point. Both these methods provide global fourth order approximation to the exact solution and does not depend on asymptotic expansion. Computations are done and results are compared with exact solutions. An order table is presented which confirms the fourth order of the method.

In chapter III, A class of Non-Linear Two-Point Singularly Perturbed Boundary Value Problems is Considered. The original Boundary value Problem is reduced to an asymptotically equivalent initial value problem by an initial value technique and is then solved in the inner region by cubic spline method for inner region employed in Method 2 developed in chapter 2. The solution of the reduced problem of the original boundary value problem provides the solution in the outer region. Computations are done and the results are compared with solution obtained by others. Order table is presented for each example which gives the computational rate of convergence obtained by double mesh principle.

In chapter IV, A class of general linear singularly perturbed boundary value problems is considered. We first consider the problem without first derivative term. Ahlberg's cubic spline in terms of moments is considered. A second order variable mesh finite difference scheme is developed using the continuity conditions of second derivatives of the spline at the interior nodes. Then the

variable mesh scheme is generalised for the problems with first derivative terms by taking some second order approximations.

In Chapter V, A class of general semi-linear singularly perturbed boundary value problems is considered. We first considered the corresponding linear singularly perturbed problem. A family of variable mesh methods based on cubic spline approximations is developed, which gives third order approximations to the solution of the linear problem. Then these methods are used for solving semi-linear problems using a quasi-linearisation technique. When a parameter namely mesh ratio parameter present in the methods is taken to be unity, the family of third order methods reduces to a family of fourth order methods, and in addition, if other parameter introduced in the method is also taken to be unity, the family of methods reduces to a well known Numerov's method for the corresponding regular problem.

In Chapter VI, the variable mesh methods based on cubic spline approximations developed in the previous chapter are generalised for the general nonlinear singularly perturbed boundary value problems.

In each chapter, error analysis is given and computations are done by taking some well known examples. All computations are done on HP9000 supermini computer at IIT Kanpur in double precision.

CHAPTER II

Cubic Spline Methods for Singularly Perturbed Initial Value Problems

2.1 INTRODUCTION :

In many areas of application, notably fluid mechanics, electrical networks, chemical reactions and control theory, singularly perturbed initial value problems arise which have a narrow region known as 'initial layer' near the initial point where solution changes very rapidly.

The use of conventional numerical methods for singularly perturbed initial value problems require a very fine mesh in the initial layer region, which makes these methods quite demanding on the computer time. Also the problem may become ill-posed numerically when mesh size gets too small. The use of asymptotic methods such as 'Matched Asymptotic Expansion', which consists of (a) dividing the problem into an inner region problem and outer region problem, (b) expressing the inner and outer solutions as asymptotic expansions, (c) equating various terms in the inner and outer solutions to determine the various coefficients and (d) combining the inner and outer solutions in some fashion to obtain a uniformly valid solution, is not a routine exercise but requires skill and experimentation efforts.

The object of this chapter is to present efficient numerical methods using cubic spline which give fourth order approximate

solution globally for singularly perturbed initial value problems (SPIVP) along with the smooth second order derivatives of the solution. These methods do not depend on asymptotic expansion and also do not require very fine mesh.

First consider the Linear Singularly Perturbed Initial Value Problem (LSPIVP)

$$\varepsilon y' + p(x,\varepsilon)y = q(x,\varepsilon) \qquad a \le x \le b \tag{2.1}$$

$$y(a) = \alpha \tag{2.2}$$

where ε is a small positive parameter, $0 < \varepsilon \ll 1$, α is a given constant, $p(x,\varepsilon) > 0$ and $q(x,\varepsilon)$ are sufficiently smooth functions. Under these assumptions, (2.1)-(2.2) has a unique solution (Keller 1968). Also it is expected that the solution of (2.1)-(2.2) would converge to the limiting solution $u(x)$ as $\varepsilon \longrightarrow 0$ where

$$u(x) = \frac{q(x,0)}{p(x,0)} \tag{2.3}$$

is the solution of the reduced problem at least away from an initial boundary layer region of non-uniform convergence.

Then the method is modified and generalized for the Non-Linear Singularly Perturbed Initial Value Problem (NLSPIVP).

$$\varepsilon y' = f(x,y,\varepsilon) \qquad a \le x \le b \tag{2.4}$$

$$y(a) = \alpha \tag{2.5}$$

Where $\frac{\partial f}{\partial y}(x,y,0)$ is non-zero on (a,b) and has only one stable root for all values of x and y and there is an initial layer near $x = a$. Then the reduced problem

$$f(x,u,0) = 0, \tag{2.6}$$

will provide the limiting solution $u_0(x)$ and also it is expected that the solution of (2.4)-(2.5) would converge to the this solution of (2.6) as $\varepsilon \longrightarrow 0$ at least away from an initial boundary layer region of non-uniform convergence. Since $u_0(a) \neq y(a)$, the solution generally converges non-uniformly at $x = a$.

In this chapter, numerical methods using a fourth order cubic spline for SPIVP's are presented. Our methods consists of:

(a) Obtaining a solution of the reduced problem which provides the limiting solution;

(b) Dividing the interval into an inner and an outer region;

(c) Obtaining an $O(h^4)$ approximation in the inner region using a cubic spline;

(d) Obtaining a global approximation on the interval [a,b] by combining the solutions of inner and outer region problems.

These methods are analyzed and it is shown that they give $O(h^4)$ convergence in the stretched variable over the inner region.

2.2 METHOD 1 (For LSPIVP's)

Using the stretching of the independent variable x by taking $t = x/\varepsilon$, we transform LSPIVP (2.1)-(2.2) to the following initial value problem

$$\frac{dz}{dt} + p(t,\varepsilon)z = q(t,\varepsilon) \tag{2.7}$$

$$z(t_0) = \alpha, \tag{2.8}$$

where $t_0 = a/\varepsilon$.

2.2.1 CUBIC SPLINE METHOD FOR INNER REGION PROBLEM :

Consider (2.7)-(2.8) on inner region $[a/\varepsilon,\ a/\varepsilon + T]$, where T is a positive integer. Given the values $z_0, z_1, \ldots, z_N$ of the

function $z(t)$ at the equally spaced nodal points $a/\varepsilon = t_0$, $t_1, \ldots, t_{N-1}, t_N = a/\varepsilon + T$, there exists an interpolating cubic spline $S(t)$ with the following properties:

(i) $S(t)$ is a polynomial of degree three on $[t_i, t_{i+1}]; i = 0(1)N-1$.

(ii) $S(t)$ is $C^2[t_0, t_N]$.

(iii) $S(t_i) = z_i$, $i = 0(1)N$.

Consider the cubic spline given by (Ahlberg et al. 1967)

$$S(t) = m_i \frac{(t_{i+1}-t)^2(t-t_i)}{h^2} - m_{i+1} \frac{(t-t_i)^2(t_{i+1}-t)}{h^2}$$

$$+ z_i \frac{(t_{i+1}-t)^2[2(t-t_i)+h]}{h^3} + z_{i+1} \frac{(t-t_i)^2[2(t_{i+1}-t)+h]}{h^3}$$

where $m_i = S'(t_i)$

$$t_i \le t \le t_{i+1}, \quad i = 0(1)N-1 \tag{2.9}$$

and the second derivative is given by

$$S''(t) = -2m_i \frac{2t_{i+1} + t_i - 3t}{h^2} - 2m_{i+1} \frac{t_{i+1} + 2t_i - 3t}{h^2}$$

$$+ 6 \frac{z_{i+1} - z_i}{h^3} (t_{i+1} + t_i - 2t). \tag{2.10}$$

Now, the limiting values from the two sides are

$$S''(t_i-) = \frac{2m_{i-1}}{h} + \frac{4m_i}{h} - 6 \frac{z_i - z_{i-1}}{h^2} \tag{2.11}$$

$$S''(t_i+) = - \frac{4m_i}{h} - \frac{2m_{i+1}}{h} + 6 \frac{z_{i+1} - z_i}{h^2} \tag{2.12}$$

The continuity condition imposed on interior nodes $t_i(i=1,\ldots,N-1)$ gives

$$m_{i-1} + 4m_i + m_{i+1} = \frac{3}{h}(z_{i+1} - z_{i-1}) \quad , \quad i= 1(1)\ N-1. \qquad (2.13)$$

Putting

$$m_i = z_i' = -p(t_i,\varepsilon)z_i + q(t_i,\varepsilon) \qquad i = 0(1)N \qquad (2.14)$$

in Eq.(2.13), we get the following two-step scheme for problem (2.7)-(2.8):

$$(\frac{3}{h} - p_{i-1})z_{i-1} - 4p_iz_i - (\frac{3}{h} + p_{i+1})z_{i+1} = -(q_{i-1} + 4q_i + q_{i+1})$$

$$i = 1(1)N-1 \qquad (2.15)$$

with $z_0 = \alpha$.

which is fourth order Milne's scheme for the linear initial value problem.

We next present our cubic spline method for solving (2.7)-(2.8) on $t_0 \le t \le t_N$.

Step 1 : Using the above scheme, approximate solutions $z_0^*, z_1^*, \ldots, z_N^*$, at the nodal points are obtained.

Step 2 : With $z_0^*, z_1^*, \ldots, z_N^*$, obtained in step 1, we compute $m_1^*, \ldots, m_N^*$, as

$$m_i^* = -p(t_i,\varepsilon)z_i^* + q(t_i,\varepsilon) \qquad i=1(1)N \qquad (2.16)$$

with

$$m_0^* = -p(t_0,\varepsilon)\alpha + q(t_0,\varepsilon)$$

Step 3 : With the values of $z_0^*, z_1^*, \ldots, z_N^*$ and $m_0^*, m_1^*, \ldots, m_N^*$ as obtained above, we construct the cubic spline $s^*(t)$ for (2.7)-(2.8) on $t_0 \le t \le t_N$ by

$$s^*(t) = m_i^* \frac{(t_{i+1}-t)^2(t-t_i)}{h^2} - m_{i+1}^* \frac{(t-t_i)^2(t_{i+1}-t)}{h^2}$$

$$+ z_i^* \frac{(t_{i+1}-t)^2[2(t-t_i)+h]}{h^3} + z_{i+1}^* \frac{(t-t_i)^2[2(t_{i+1}-t)+h]}{h^3}$$

$$t_i \le t \le t_{i+1}, \qquad i = 0(1)N-1 \tag{2.17}$$

2.2.2 <u>REDUCED PROBLEM</u> :

Solve (2.3) to get the solution $u_0(x)$ which is the solution of (2.1)-(2.2) in the outer region.

Now finally we have global approximate solution $y^*(x)$ of (2.1)-(2.2) on [a,b] as

$$y^*(x) = \begin{cases} s^*(x/\varepsilon) & \text{for } x \in [a, T\varepsilon] \\ u_0(x) & \text{for } x \in (T\varepsilon, b] \end{cases} \tag{2.18}$$

Note: In this method, there are two things which need some modifications. First, the cutting point, i.e. the point of division of inner and outer region, is arbitrary and secondly, the solution is discontinuous at this point. In the next section we have taken care of these points and also generalized the method for NLSPIVP's.

2.3 METHOD 2 (For NLSPIVP's)

Using the stretching of the independent variable x by taking $t = x/\varepsilon$, we transform the NLSPIVP (2.4)-(2.5) to the following initial value problem

$$\frac{dz}{dt} = f(t,z(t),\varepsilon) \tag{2.19}$$

$$z(t_0) = \alpha \tag{2.20}$$

where $t_0=a/\varepsilon$.

2.3.1 REDUCED PROBLEM :

Solve the reduced problem (2.6) and select the stable root $u_0(x)$ i.e., for which $\frac{\partial f}{\partial y}(x,u_0(x))$ is strictly negative. Then this root will provide the limiting solution of (2.4)-(2.5).

2.3.2 CUTTING POINT TECHNIQUE :

Let $t_p=t_0+1$ be the initial cutting point.

For a positive integer N (not large), let $h = (t_p-t_0)/N$,

$$t_k = t_0+kh,\ k=0(1)N\ , \qquad t_N = t_p\ .$$

Step 1: For the initial value problem (2.19)-(2.20) over the interval $[t_0,t_p]$, obtain the approximate solution z_i^*, $i = 0(1)N$ at nodal points t_k, $k = 0(1)N$, by classical fourth order Runge-Kutta method (In fact, any fourth order method can be applied).

Step 2: If $|z_N^* - u_0(t_p)| \le h^3$, (2.21)

then we stop and take t_p to be the desired cutting point, else we increase t_p by 1 and N by 2N then repeat step 1 until (2.21) is satisfied.

Thus, we have now obtained the desired value of the cutting point t_p and an approximation to the solution at the nodal points

over the inner region $[t_0, t_p]$. Since this solution z_i^*, $i = 0(1)N$, is only a crude approximation, to obtain the solution (global) at the non-nodal points along with the smooth second derivatives in the inner region, we do the following:

2.3.3 CUBIC SPLINE METHOD FOR INNER REGION PROBLEM :

We next present our cubic spline method for solving (2.19)-(2.20) on $t_0 \le t \le t_p$.

Step 1 : With $z_0^*, z_1^*, \ldots, z_{N-1}^*$, obtained in the section 2.3.2, we compute $m_1^*, \ldots, m_{N-1}^*$ from the 'conditions of continuity'

$$\left.\begin{aligned} m_{i-1}^* + 4m_i^* + m_{i+1}^* &= \frac{3}{h}\,(z_{i+1}^* - z_{i-1}^*)\ , \quad i = 1(1)N-2 \\ m_{N-2}^* + 4m_{N-1}^* + m_N^* &= \frac{3}{h}\,(u_0(t_N) - z_{N-2}^*) \end{aligned}\right\} \qquad (2.22)$$

with

$$m_0^* = f(t_0, \alpha, \varepsilon), \qquad m_N^* = f(t_N, u_0(t_N), \varepsilon)$$

Since, it is cumbersome to write each time a separate equation for replacing z_N^* by $u_0(t_N)$, therefore, now onwards, in this section we take $z_N^* = u_0(t_N)$.

Step 2 : With the values of $z_0^*, z_1^*, \ldots, z_N^*$ and $m_0^*, m_1^*, \ldots, m_N^*$ as obtained above, we construct the cubic spline $S^*(t)$ for (2.19)-(2.20) on $t_0 \le t \le t_p$ using (2.17) of Method 1.

Now finally we have global approximate solution $y^*(x)$ of (2.4)-(2.5) on [a,b] as

$$y^*(x) = \begin{cases} S^*(x/\varepsilon) & \text{for } x \in [a,x_p] \\ u_0(x) & \text{for } x \in [x_p,b] \end{cases} \tag{2.23}$$

where $x_p = t_p\varepsilon$

Note : Since $S^*(x_p/\varepsilon) = u_0(t_p)$, the solution is continuous at the cutting point. In fact, its first derivative is also continuous at this point.

2.4 ERROR ANALYSIS :

Let $e(t)$ denote the error in the spline solution $S^*(t)$ for the solution of (2.7)-(2.8) and (2.19)-(2.20) (Since the error analysis is more or less common for both methods, it is given together).

i.e., $e(t) = z(t)-S^*(t) = [z(t) - S(t)] + [S(t) - S^*(t)]$ (2.24)

$= e_1(t) + e_2(t)$ (2.25)

where $e_1(t)$ is the error due to spline interpolation

and $e_2(t)$ is the error due to descretization of the differential equation.

In the following, we use for $v \in C[a,b]$, $\| v \|_\infty = \max_{[a,b]} |v(t)|$.

It is well known that the cubic spline interpolation is fourth order process (Hilderbrand 1956). Therefore, for $z \in C^4[t_0,t_p]$, we have

$$\| e_1\|_\infty \le C_1h^4, \qquad C_1 \text{ a constant} \tag{2.26}$$

Next, for $e_2(t)$, from (2.9) and (2.17) we get

$$e_2(t) = (m_i - m_i^*)\frac{(t_{i+1}-t)^2(t-t_i)}{h^2} + (m_{i+1}^* - m_{i+1})\frac{(t-t_i)^2(t_{i+1}-t)}{h^2}$$

$$+ (z_i - z_i^*)\frac{(t_{i+1}-t)^2[2(t-t_i)+h]}{h^3}$$

$$+ (z_{i+1} - z_{i+1}^*)\frac{(t-t_i)^2[2(t_{i+1}-t)+h]}{h^3}$$

$$\text{for } t_i \le t \le t_{i+1}\ ,\ i=0(1)\ N-1 \qquad (2.27)$$

Let $Z = (z_1,\ldots,z_N)^t, \quad Z^* = (z_1^*,\ldots,z_N^*)^t$

$M = (m_1,\ldots,m_N)^t, \quad M^* = (m_1^*,\ldots,m_N^*)^t$

In the following, we use for $X = (x_1,\ldots,x_N)^t$,

$$\| X \|_\infty = \max_{1\le i\le N} |x_i|$$

from (2.27) we have

$$| e_2(t) | \le | M-M^* |\ f_1(t) + | Z-Z^* |\ f_2(t)$$

$$\le \| M-M^* \|_\infty f_1(t) + \| Z-Z^* \|_\infty f_2(t) \qquad (2.28)$$

where,

$$f_1(t) = \frac{(t_{i+1}-t)^2(t-t_i)}{h^2} + \frac{(t-t_i)^2(t_{i+1}-t)}{h^2}$$

and $$f_2(t) = \frac{(t_{i+1}-t)^2[2(t-t_i)+h] + (t-t_i)^2[2(t_{i+1}-t)+h]}{h^3}$$

Clearly, maxima of $f_1(t)$ and $f_2(t)$ is attained at $t = \frac{t_i+t_{i+1}}{2}$

Hence, $\max_t f_1(t) = \frac{h}{4}$ and $\max_t f_2(t) = 1$

Therefore,

$$\| e_2 \|_\infty \le \frac{h}{4} \| M-M^* \|_\infty + \| Z-Z^* \|_\infty \tag{2.29}$$

Bound for $\| M-M^* \|_\infty$ (For Method 1)

From (2.14) and (2.16), we have

$$m_i - m_i^* = -p(t_i,\varepsilon)(z_i - z_i^*)$$

$$\Rightarrow \qquad | m_i - m_i^* | = p(t_i,\varepsilon) | z_i - z_i^* |$$

$$\Rightarrow \qquad \| M-M^* \|_\infty \le P \| Z-Z^* \|_\infty \tag{2.30}$$

where $P = \max_i p(t_i,\varepsilon)$.

Bound for $\| M-M^* \|_\infty$ (For Method 2)

Writing (2.13) in the matrix-vector form, we have

$$A\ M = (3/h)\ B\ Z - C \tag{2.31}$$

Where A and B are tridiagonal matrices given by

$$A_i = i^{th} \text{ row of } A = \text{Trid } [\ 1 \quad 4 \quad 1\ ]$$

$$B_i = i^{th} \text{ row of } B = \text{Trid } [\ -1 \quad 0 \quad 1\ ] \qquad i = 1,\ldots,N$$

and $C = (m_0, 0, \ldots, 0, m_N)$ is an N-column vector.

where $m_0 = f(t_0, z_0, \varepsilon)$ and $m_N = f(t_N, z_N, \varepsilon)$

Similarly, writing (2.22) in the matrix-vector form, we have

$$A\ M^* = (3/h)\ B\ Z^* - C^* \tag{2.32}$$

where $\quad C^* = (\ m_0^*, 0, \ldots, 0, m_N^*)$ is an N-column vector.

Subtracting (2.32) from (2.31) we have

$$A\ (M - M^*) = (3/h)\ B\ (Z - Z^*) + (C^* - C) \tag{2.33}$$

From (2.33)

$$\| M-M^*\|_\infty \le (3/h)\ \|A^{-1}\|\ \|B\|\ \|Z-Z^*\|_\infty + \|A^{-1}\|\ |\ C-C^*| \tag{2.34}$$

Note that $\|A^{-1}\| \le 1/2$,also $\|B\| = 2$,

From the definition of m_N, m_N^* and the mean value theorem,

We have

$$\begin{aligned} |\ m_N - m_N^*\ | &\le d\ |\ z_N - u_0(t_p)\ | \\ &\le d\ (|\ z_N - z_N^*\ | + |\ z_N^* - u_0(t_p)\ | \\ &\le d\ (\|\ Z-Z^*\|_\infty + h^3) \qquad \text{(using (2.21)} \end{aligned} \tag{2.35}$$

where $\quad d = |\ \frac{\partial f}{\partial z}(t_N, \bar{z}_N)\ |$, $\bar{z}_N$ lying between z_N and z_N^*

Now, from the definition of C and C^*, we have

$$|\ C-C^*| = |\ m_N - m_N^*\ |$$

which implies

$$|\ C-C^*| \le d\ (\|\ Z-Z^*\|_\infty + h^3) \tag{2.36}$$

Therefore, we obtain

$$\|M - M^*\ \|_\infty \le (\ \frac{3}{h} + \frac{d}{2})\|\ Z-Z^*\|_\infty + \frac{d}{2}\ h^3 \tag{2.37}$$

Since we are using fourth order schemes in both methods for approximating z(t) at nodal points we have

$$\| Z - Z^{*} \|_{\infty} \le C_2 h^4 \tag{2.38}$$

where C_2 is a constant independent of h.

Therefore, from (2.29),(2.30) and (2.37)

$$\| e_2 \|_{\infty} \le C_3 h^4 + C_4 h^5 \tag{2.39}$$

$$\text{where } C_3 = \begin{cases} C_2 & \text{(For Method 1)} \\ \frac{7}{4} C_2 + \frac{d}{8} & \text{(For Method 2)} \end{cases}$$

$$\text{where } C_4 = \begin{cases} -\frac{P}{4} C_2 & \text{(For Method 1)} \\ \frac{d}{8} C_2 & \text{(For Method 2)} \end{cases}$$

Finally, combining (2.26) and (2.39) we have

$$\| e \|_{\infty} \le \bar{C} h^4, \qquad \bar{C} = C_1 + C_3 \tag{2.40}$$

Thus, we have proved the following:

<u>Theorem 2.1</u>: Let $z(t) \in C^4[t_0,t_p]$, then our cubic spline methods provide an order h^4 convergent approximation $S^*(t)$ for the solution z(t) of the initial value problems (2.7)-(2.8) and (2.19)-(2.20); that is,

$$\| e \|_{\infty} \le \bar{C} h^4$$

where $\bar{C}$ is a constant independent of h.

2.5 NUMERICAL EXAMPLES :

Example 2.1. (Artificial test problem of Lapidus and Seinfeld 1971, Doolan et.al 1980)

$$\varepsilon y' = g(x) + \varepsilon g'(x) - y \qquad 0 \le x \le 1$$

$$y(0) = 10, \; g(x) = 10 - (10 + x)e^{-x}$$

Here $f(x,u,0) = 0 \Rightarrow u = g(x)$ and $f_y(x,g(x),0) = -1$ which is stable. Thus, $u_0(x) = g(x)$ provides the limiting solution.

The exact solution is given by

$$y(x) = g(x) + 10\, e^{-x/\varepsilon}$$

The computational results obtained by Method 1 and Method 2 are presented in Tables 2.1 and 2.2 respectively, for different values of h and ε.

Example 2.2 (Edsberg 1976, Doolan et.al 1980)

$$\varepsilon\, y' = -2\, y^2 \qquad 0 \le x \le 1$$

$$y(0) = 1$$

Here $f(x,u,0) = 0 \Rightarrow u = 0$ and $f_y(x,0,0) = 0$, but $u = 0$ is unique Therefore it provides the limiting solution $u_0(x) = 0$.

The exact solution is given by

$$y(x) = 1 / [1 + 2\, x/\varepsilon\,]$$

The computational results obtained by Method 2 are presented in Table 2.3 for the different values of h and ε.

Example 2.3

$$\varepsilon\, y' = xy(1 + y) \qquad\qquad 1 \le x \le 2$$

$$y(1) = -2$$

Here $f(x,u,0) = 0 \Rightarrow u = 0,\ -1$ and $f_y(x,0,0) = x$ which is unstable while $f_y(x,-1,0) = \neg x$ which is stable.

Thus, $u_0(x) = -1$ provides the limiting solution.

The exact solution is given by

$$y(x) = \frac{\exp((x^2-1)/\ 2\varepsilon + \log 2\ )}{1 - \exp((x^2-1)/\ 2\varepsilon + \log 2)}$$

The computational results obtained by Method 2 are presented in Table 2.4 for different values of h and ε.

2.6 DISCUSSION :

We have presented numerical method which gives $O(h^4)$ approximation to the solution of singularly perturbed initial value problems. In the method 1(For LSPIVP's), choice of the cutting point is arbitrary and the solution at the cutting point is discontinuous. In method 2(For NLSPIVP's) which is generalization and modification of method 1, a numerical cutting point technique, which is iterative in nature, is given for selecting the cutting point. Also the continuity of the solution at cutting point is guaranteed.

Theoretically, problem (2.19)-(2.20) can be solved on interval $[a/\varepsilon, b/\varepsilon]$ by any initial value method, practically ε being very small, this interval is quite large. It is observed from the computation that by the cutting point technique, the effective size of the inner region in the stretched variable

remain almost same with the deceasing value of ε.

Also Milne's method can be used for obtaining solution at nodal points in method 2 and m's can be obtained subsequently as in method 1, but for solution to be continuous at the cutting point we have to change solution at this point from z_N^* to $u_0(t_p)$, causing the violation of continuity conditions. In order to make solution smooth in the inner region along with matching it with outer solution at the cutting point, we have to recompute m's. Therefore solution at the nodal points serves only as fourth order approximations, which can be provided by any other fourth order method also. We preferred to use classical Runge-Kutta method for this purpose.

Use of conventional methods for singularly perturbed initial value problems by transferring ε to the right hand side generally require the step size to satisfy $h \ll \varepsilon$ even in the region away from the boundary layer region and thus restriction on the step size makes these methods quite demanding on the computer time. Our method does not involve any constraint on the step size h.

Three test examples have been solved by this method. Example no.1 is solved by method 1 and method 2 for comparison. We have computed solutions at the middle of the nodal points and we have tabulated the maximum and minimum error in both inner and outer regions along with the absolute difference(ds) between computed inner and outer computed solution at the cutting point for different values of h and ε, it is observed that ds is non zero when we apply method 1 while it is zero for method 2, which confirms the continuity of the solution at the cutting point for

method 2. Also the rate of convergence of these approximations which confirms the fourth order of our cubic spline methods in the inner region is given in table 2.5. From the tables it is observed that present methods approximates the exact solution well in both inner and outer regions and also the error does not grow with the decreasing values of ε. In fact by our methods, which gives a global approximation of the solution, approximation at any point of the interval can be obtained without any further computation.

We define the computational order of convergence for two successive values of h in the usual way i.e, for h and h/2 with respect to errors I_{max}^{h} and $I_{max}^{h/2}$:

$$\text{Order} = \frac{\left[\ \log\left(I_{max}^{h}\right) - \log\left(I_{max}^{h/2}\right)\ \right]}{\log 2}$$

Heading in the tables are as follows:

h	Mesh size
I_{max}	Maximum error in the inner region
I_{min}	Minimum error in the inner region
O_{max}	Maximum error in the outer region
O_{min}	Minimum error in the outer region
ds	Absolute difference between computed inner and outer solution at the cutting point

Table 2.1(a)

Example 2.1, $\varepsilon = 10^{-3}$ (Method 1)

h		T = 2	T = 5	T = 10
1/5	I_{max}	4.28810 E-5	4.28810 E-5	4.28810 E-5
	I_{min}	2.41803 E-5	6.01878 E-7	6.01891 E-7
	O_{max}	1.35335	6.73794 E-2	4.53400 E-4
	O_{min}	0.0	0.0	0.0
	ds	1.35339	6.74890 E-2	1.38547 E-4
1/10	I_{max}	2.90740 E-6	2.90740 E-6	2.90740 E-6
	I_{min}	1.86229 E-6	8.27880 E-2	1.18289 E-10
	O_{max}	1.35335	6.73794 E-2	4.53400 E-4
	O_{min}	0.0	0.0	0.0
	ds	1.35339	6.73756 E-2	4.34932 E-4
1/20	I_{max}	1.92246 E-7	1.92246 E-7	1.92246 E-7
	I_{min}	1.16134 E-7	1.12907 E-8	1.61059 E-11
	O_{max}	1.35335	6.73794 E-2	4.53400 E-4
	O_{min}	0.0	0.0	0.0
	ds	1.35339	6.73756 E-2	4.34932 E-4

Table 2.1(b)

Example 2.1, $\varepsilon = 10^{-5}$ (Method 1)

h		T = 2	T = 5	T = 10
1/5	I_{max}	4.28810 E-5	4.28810 E-5	4.28810 E-5
	I_{min}	2.41803 E-5	6.01878 E-7	6.01891 E-7
	O_{max}	1.35335	6.73794 E-2	4.53400 E-4
	O_{min}	0.0	0.0	0.0
	ds	1.35339	6.74890 E-2	1.38547 E-4
1/10	I_{max}	2.90740 E-6	2.90740 E-6	2.90740 E-6
	I_{min}	1.86229 E-6	8.27880 E-2	1.18289 E-10
	O_{max}	1.35335	6.73794 E-2	4.53400 E-4
	O_{min}	0.0	0.0	0.0
	ds	1.35339	6.73756 E-2	4.34932 E-4
1/20	I_{max}	1.92246 E-7	1.92246 E-7	1.92246 E-7
	I_{min}	1.16134 E-7	1.12907 E-8	1.61059 E-11
	O_{max}	1.35335	6.73794 E-2	4.53400 E-4
	O_{min}	0.0	0.0	0.0
	ds	1.35339	6.73756 E-2	4.34932 E-4

Table 2.2

Example 2.1 (Method 2)			
h		$\varepsilon = 10^{-3}$ $x_p = 2E-2$	$\varepsilon = 10^{-5}$ $x_p = 2E-4$
1/5	I_{max}	4.45836 E -5	4.45836 E -5
	I_{min}	6.32425 E -11	7.82276 E -11
	O_{max}	2.06115 E -8	2.06115 E -8
	O_{min}	0.0	0.0
	ds	0.0	0.0
1/10	I_{max}	2.49986 E -6	2.49986 E -6
	I_{min}	3.41245 E -11	4.27864 E -11
	O_{max}	2.06115 E -8	2.06115 E -8
	O_{min}	0.0	0.0
	ds	0.0	0.0
1/20	I_{max}	1.52456 E -7	1.52456 E -7
	I_{min}	1.45095 E -12	2.49720 E -12
	O_{max}	2.06115 E -8	2.06115 E -8
	O_{min}	0.0	0.0
	ds	0.0	0.0

Table 2.3

h		$\varepsilon = 10^{-3}$, x_p = 1.2E-2	$\varepsilon = 10^{-5}$, x_p = 1.2E-4
1/5	I_{max}	8.65242 E -4	8.65242 E -4
	I_{min}	3.48361 E -9	3.48361 E -9
	O_{max}	2.73582 E -6	2.73582 E -6
	O_{min}	0.0	0.0
	ds	0.0	0.0
1/10	I_{max}	7.46802 E -5	7.46802 E -5
	I_{min}	3.50942 E -10	3.50942 E -10
	O_{max}	2.73582 E -6	2.73582 E -6
	O_{min}	0.0	0.0
	ds	0.0	0.0
1/20	I_{max}	5.48070 E -6	5.48070 E -6
	I_{min}	2.34987 E -11	2.34987 E -11
	O_{max}	2.73582 E -6	2.73582 E -6
	O_{min}	0.0	0.0
	ds	0.0	0.0

Example 2.2 (Method 2)

Table 2.4

Example 2.3 (Method 2)			
h		$\varepsilon = 10^{-3}$ $x_p = 1+13E-2$	$\varepsilon = 10^{-5}$ $x_p = 1+13E-4$
1/5	I_{max}	3.89349 E -4	3.89755 E -4
	I_{min}	0.0	0.0
	O_{max}	1.12921 E -6	1.12921 E -6
	O_{min}	0.0	0.0
	ds	0.0	0.0
1/10	I_{max}	3.14718 E -4	3.15041 E -4
	I_{min}	0.0	0.0
	O_{max}	1.129211 E -6	1.12921 E -6
	O_{min}	0.0	0.0
	ds	0.0	0.0
1/20	I_{max}	2.21353 E -4	2.21556 E -4
	I_{min}	0.0	0.0
	O_{max}	1.129211 E -6	1.12921 E -6
	O_{min}	0.0	0.0
	ds	0.0	0.0

Table 2.5

	h	I_{max}	Order
Example 2.1 (Method 1)	1/5	4.28810 E -5	
	1/10	2.90740 E -6	3.88
	1/20	1.92246 E -7	3.92
Example 2.1 (Method 2)	1/5	4.45836 E -5	
	1/10	2.49985 E -6	4.16
	1/20	1.52456 E -7	4.03
Example 2.2 (Method 2)	1/5	8.65242 E -4	
	1/10	7.46802 E -5	3.53
	1/20	5.48070 E -6	3.77
Example 2.3 (Method 2)	1/5	3.89349 E -4	
	1/10	3.14718 E -5	3.63
	1/20	2.21355 E -6	3.83

CHAPTER III

On a Class of Non-Linear Singularly Perturbed Boundary Value Problems Via Initial Value Methods

3.1 INTRODUCTION :

In general, finding numerical solution of a boundary value problem is more difficult than finding numerical solution of corresponding initial value problem. Therefore it is better to convert the second order problem into an asymptotically equivalent first order problem, wherever possible. Solving non-linear boundary value problems directly by finite difference schemes requires linearization. An iterative process with a fine initial guess and good number of iterations is then required to get an acceptable solution.

In this chapter, we have given an alternative approach for solving a class of non-linear singularly perturbed boundary value problems, so that the above complexities can be avoided. Solving these problems by asymptotic methods is also far from trivial since it requires lot of experimental skill. We have converted a class of singularly perturbed non-linear boundary value problems into asymptotically equivalent singularly perturbed initial value problems. The cutting point technique is then applied to separate the inner and outer regions. The solution in the inner region is globally approximated by fourth order cubic spline method for

singularly perturbed initial value method, developed in the previous chapter. The solution of the reduced problem is taken as the solution in the outer region

Consider a class of non-linear singular perturbation problem of the form

$$\varepsilon y''(x) + [p(y(x))]' + q(x,y(x)) = r(x) \qquad a \le x \le b \qquad (3.1)$$

$$y(a) = \alpha, \qquad y(b) = \beta; \qquad (3.2)$$

here, ε is a small parameter, $0 < \varepsilon << 1$; α and β are given constants; $p(y)$, $q(x,y)$, $r(x)$ are assumed to be sufficiently differentiable functions. Furthermore, we assume that the problem (3.1)-(3.2) has a solution which displays a boundary layer of width $o(\varepsilon)$ at $x = a$ for small value of ε.

This problem has been considered earlier by Kadalbajoo and Reddy(1988). After reducing the original problem to an asymptotically equivalent singularly perturbed initial value problem, they have applied classical fourth order Runge-Kutta method. But since the resulting problem is still singularly perturbed, applying any conventional method for solving it, requires a very fine mesh ($h << \varepsilon$) to satisfy the criterion of absolute stability for obtaining an acceptable approximate solution. In view of ε being very small in practice, the restriction on the step size h would often lead to prohibitive cost of computational time.

3.2 INITIAL VALUE METHOD :

We describe the technique in a step-wise manner as follows:

Step 1. Obtain the reduced problem by setting $\varepsilon = 0$ in (3.1) and solve it for the solution. Let $y_o(x)$ be the solution of the reduced problem of (3.1)-(3.2), that is

$$[p(y_o(x))]' + q(x,y_o(x)) = r(x) \tag{3.3}$$

$$y_o(b) = \beta \tag{3.4}$$

Step 2 Set up the approximate equation to Eq. (3.1) as follows:

$$\varepsilon y''(x) + [p(y(x))]' + q(x,y_o(x)) = r(x) \tag{3.5}$$

where the $y(x)$ term in $q(x,y(x))$ is replaced by $y_o(x)$, the solution of the reduced problem (3.3)-(3.4).

Step 3. Replace the approximated second-order problem (3.5) by an asymptotically equivalent first order problem as follows:

By integrating (3.5), we obtain

$$\varepsilon y'(x) + p(y(x)) + Q(x) = R(x) + K \tag{3.6}$$

where

$$Q(x) = \int q(x,y_o(x))dx,$$

$$R(x) = \int r(x)dx$$

and K is a constant to be determined.

In order to determine the constant K, the condition that the reduced equation of Eq.(3.6) should satisfy the boundary condition $y(b) = \beta$ gives

$$p(y(b)) + Q(b) = R(b) + K \tag{3.7}$$

so that

$$K = p(\beta) + Q(b) - R(b) \tag{3.8}$$

Now, we adjoin the condition $y(a) = \alpha$ to (3.6) to obtain a first-order problem as follows:

$$\varepsilon y'(x) + p(y(x)) + Q(x) = R(x) + K \tag{3.9}$$

$$y(a) = \alpha \tag{3.10}$$

where K is a constant given by Eq.(3.8).

Step 4. Setting $t = x/\varepsilon$ in (3.9) and rescaling with

$$y(x) = z(t) \tag{3.11}$$

$$y'(x) = z'(t)/\varepsilon \tag{3.12}$$

We obtain the resulting problem as

$$z'(t) + p(z(t)) + Q(t) = R(t) + K \tag{3.13}$$

$$z(a/\varepsilon) = \alpha \tag{3.14}$$

Now, we apply the cutting point technique(Sec 2.3.2, Chap 2) for finding the cutting point t_p, then the cubic spline method for finding the solution of the inner region problem(Sec 2.3.3, Chap 2) is applied and solution y_0 of the reduced problem is taken as the solution of the outer region problem.

Step 5 Finally, the solutions of the inner and outer region we have global approximate solution

$$y^*(x) = \begin{cases} S^*(x/\varepsilon) & \text{for } x \in [a, x_p] \\ y_0(x) & \text{for } x \in [x_p, b] \end{cases} \quad (3.15)$$

where $x_p = \varepsilon t_p$ and $S^*(x/\varepsilon)$ is the cubic spline solution obtained from Method 2 of second chapter.

3.3 ERROR ANALYSIS :

Since, after reducing the original boundary value to an initial value problem and then applying the method 2 developed in the previous chapter, the error analysis of the present method is similar to that of above said method, therefore, in order to avoid repetition, we give only brief description of the error propagation.

Let e(t) denote the error in the spline solution $S^*(t)$ for the solution of (3.13)-(3.14), in the inner region $[a_{0/\varepsilon}, t_p]$, i.e.,

$$e(t) = z(t) - S^*(t) = [z(t) - S(t)] + [S(t) - S^*(t)] \quad (3.16)$$

$$= e_1(t) + e_2(t) \quad (3.17)$$

where $e_1(t)$ is the error due to spline interpolation

and $e_2(t)$ is the error due to descretization of the differential equation.

Also, from Eq.(2.26) and (2.29) of Chap 2, we have

$$\| e_1 \|_\infty \le C_1 h^4, \qquad C_1 \text{ a constant} \quad (3.18)$$

$$\| e_2 \|_\infty \le \frac{h}{4} \| M - M^* \|_\infty + \| Z - Z^* \|_\infty \quad (3.19)$$

where $M = (m_1,\ldots,m_N)^t$, $M^* = (m_1^*,\ldots,m_N^*)^t$,

$Z = (z_1,\ldots,z_N)^t$, $Z^* = (z_1^*,\ldots,z_N^*)^t$ are the actual and approximate solutions of (3.13) and (3.14) at the nodal points respectively. Also,

$$C = (m_0, 0, \ldots, 0, m_N)^t, \quad C^* = (m_0^*, 0, \ldots, 0, m_N^*)^t$$

Where $m_0 = m_0^* = -p(\alpha) - Q(a_0/\varepsilon) + R(a_0/\varepsilon) + K,$ (3.20)

$$m_N = -p(z(t_N)) - Q(t_N) + R(t_N) + K \tag{3.21}$$

$$m_N^* = -p(y_0(t_N)) - Q(t_N) + R(t_N) + K \tag{3.22}$$

Therefore, Using mean value theorem, we have

$$| m_N - m_N^* | \le d\ (\| Z-Z^*\|_\infty + h^3) \tag{3.23}$$

where $d = | \frac{\partial p}{\partial z}(t_N, \bar{z}_N) |$, $\bar{z}_N$ lying between z_N and z_N^*

Now, from the definition of C and C^*, we have

$$| C-C^*| \le d\ (\| Z-Z^*\|_\infty + h^3) \tag{3.24}$$

Also from Eq(2.36) of Chap 2, we have

$$\| M-M^*\|_\infty \le (3/h)\ \|A^{-1}\|\ \|B\|\ \|Z-Z^*\|_\infty + \|A^{-1}\|\ | C-C^*| \tag{3.25}$$

where, A and B are triangular matrices with $\|A^{-1}\| \le \frac{1}{2}$ and $\|B\| = 2$

Therefore, we obtain

$$\|M - M^*\|_\infty \le \left(\frac{3}{h} + \frac{d}{2}\right) \| Z-Z^*\|_\infty + \frac{d}{2} h^3 \tag{3.26}$$

Since we are using fourth order schemes in both methods for

approximating z(t) at nodal points we have

$$\| Z - Z^* \|_\infty \le C_2 h^4 \tag{3.27}$$

where C_2 is a constant independent of h.

Therefore, from (3.19), (3.26) and (3.27)

$$\| e_2 \|_\infty \le C_3 h^4 + C_4 h^5 \tag{3.28}$$

where $C_3 = \frac{7}{4} C_2 + \frac{d}{8}$ and $C_4 = \frac{d}{8} C_2$

Finally, combining (3.18) and (3.28) we have

$$\| e \|_\infty \le \bar{C} h^4, \qquad \bar{C} = C_1 + C_2 \tag{3.29}$$

Thus, we have proved the following:

<u>Theorem 3.1</u>: Let $z(t) \in C^4[a_0/\varepsilon, t_p]$, then our cubic spline method provides an order h^4 convergent approximation $S^*(t)$ for the solution z(t) of the initial value problem (3.13)-(3.28); that is,

$$\| e \|_\infty \le \bar{C} h^4$$

where $\bar{C}$ is a constant independent of h.

3.4 NUMERICAL EXAMPLES :

<u>Example 3.1</u> Consider the following example (O'Malley (1974), page 9, Eq. (1.10), Case 2)

$$\varepsilon y'' = yy' \qquad -1 \le x \le 1$$

$$y(-1) = 0, \qquad y(1) = -1$$

We have chosen O'Malley's approximate solution for comparison,

$$y(x) = -\frac{1 - \exp\left(-\frac{(x+1)}{\varepsilon}\right)}{1 + \exp\left(-\frac{(x+1)}{\varepsilon}\right)}$$

For this example, we have a boundary layer of width $O(\varepsilon)$ at $x = -1$. (O'Malley (1974) pages 9 & 10)

Rewriting the differential equation in the form (3.1), we have

$$\varepsilon y'' - \left[\frac{y^2(x)}{2}\right]' = 0$$

Integrating this, we get

$$\varepsilon y'(x) - \frac{y^2(x)}{2} = K$$

The constant K is determined using (3.8) as

$$K = -\frac{y^2(1)}{2} = -\frac{1}{2}$$

Hence the initial value problem is given by

$$\varepsilon y' - \left[\frac{y(x)^2-1}{2}\right] = 0 \qquad -1 \le x \le 1$$

$$y(-1) = 0$$

This IVP is solved over the inner region in the stretched variable by the cubic spline method and the outer solution is given by the solution of the reduced problem

$$Y_0(x)\, Y_0'(x) = 0$$

$$Y_0(1) = -1$$

which gives $Y_0(x) = -1$.

The computational results are presented in Tables 3.1 for different values of h and ε.

Example 3.2 Consider the following example (Kevorkian and Cole (1981) page 56, Eq. (2.5.1)):

$$\varepsilon y'' + yy' - y = 0 \qquad 0 \le x \le 1$$

$$y(0) = -1, \qquad y(1) = 3.9995$$

We have chosen to use Kevorkian and Cole's uniformly valid approximation for comparison:

$$y(x) = x + C_1 \tanh\left[\left(\frac{C_1}{2}\right)\left(\frac{x}{\varepsilon} + C_2\right)\right]$$

where $C_1 = 2.9995$,

and $\quad C_2 = \frac{1}{C_1} \log\left(\frac{C_1 - 1}{C_1 + 1}\right)$

For this example we have a boundary layer of width $O(\varepsilon)$ at $x = 0$. (Kevorkian and cole 1981) page 56-66.

Rewriting the differential equation in the form of (3.1), we have

$$\varepsilon y'' + \left[\frac{y^2(x)}{2}\right]' - y(x) = 0$$

The reduced equation is given by

$$\left[\frac{y^2(x)}{2}\right]' - y(x) = 0$$

This implies two equations

$$y(x) = 0$$

$$y'(x) = 1$$

Second equation is the correct differential equation, since first equation is not satisfied at the boundary point $x = 1$.

Hence the reduced problem is given by

$$y_0'(x) = 1$$

$$y_0(x) = 3.9995$$

whose solution is

$$y_0(x) = x + 2.9995$$

We get the approximate equation as

$$\varepsilon y''(x) + \left[\frac{y^2(x)}{2} \right]' - (x + 2.9995) = 0$$

Now by integrating this, we get

$$\varepsilon y' + \left[\frac{y^2(x)}{2} \right]' - \left(\frac{x^2}{2} + (\beta-1)x \right) = K$$

where $\beta = 3.9995$

The constant K is determined using (3.8) as

$$K = \frac{y^2(1)}{2} - \left(\frac{1}{2} + \beta - 1 \right) = \frac{(\beta-1)^2}{2}$$

Hence the initial value problem is given by

$$\varepsilon y'(x) + \left[\frac{y(x)^2 - (x+\beta-1)^2}{2} \right] = 0$$

$$y(0) = -1$$

This IVP is solved in the inner region in the stretched variable by the cubic spline method.

The computational results are presented in Table 3.2 for different values of h and ε.

Example 3.3: Consider the following example [Bender and Orszag (1978) page 463, Eq. (9.7.1)]:

$$\varepsilon y'' + 2y' + e^{y} = 0 \qquad 0 \le x \le 1$$

$$y(0) = 0, \qquad y(1) = 0$$

We have chosen to use Bender and Orszag's uniformly valid approximation for comparison.

$$y(x) = \log\left[\frac{2}{1+x}\right] - \exp\left(\frac{-2x}{\varepsilon}\right)\log 2$$

For this example we have a boundary layer at x = 0.

The reduced problem is given by

$$2y_0'(x) + \exp[y_0(x)] = 0 \qquad 0 \le x \le 1$$

$$y_0(1) = 0$$

which gives

$$y_0(x) = \log\frac{2}{1+x}$$

We get the approximate equation as

$$\varepsilon y''(x) + [2y(x)]' + \frac{2}{1+x} = 0$$

Now by integrating this, we have

$$\varepsilon y'(x) + 2y(x) + 2\log(1+x) = K$$

The constant K is determined using (3.8) as

$$K = 2y(1) + 2\log(1+1) = 2\log 2$$

Hence the initial value is given by

$$\varepsilon y'(x) + 2\left[y(x) - \log\left(\frac{2}{1+x}\right)\right] = 0$$

$$y(0) = 0$$

This IVP is solved in the inner region in the stretched variable by the cubic spline and the outer solution is given by the solution of the reduced problem.

The computational results are presented in Tables 3.3 for different values of h and ε.

3.5 DISCUSSION :

We have implemented the present method on three examples by taking different values of ε and have compared the computational results with the approximate solutions developed by others. It can be observed that our method does not require a fine mesh and also does not impose any restriction on the step size h. In fact, numerical solution obtained for a crude mesh size h = 1/5 compare very well with the approximate solutions. Our method provides an alternative approach to the conventional way of solving certain class of non-linear singularly perturbed problems. The method does not require any asymptotic analysis and does not require any non linear equations to be solved.

Since the exact solutions of the test examples are not available, we used the double mesh principle defined below for finding the computational rate of convergence:

Let

$$Z_h = \max_j \mid u_j^h - u_j^{h/2} \mid , \qquad j = 0,\ldots,N-1$$

where u_j^h is the computed solution at the point $t = t_j + \lambda h$, $\lambda \in [0,1]$, on the mesh $\left\{ t_j \right\}_0^N$, where $t_j = t_{j-1} + h$, $j = 1(1)N$, and $u_j^{h/2}$ is the computed solution at the same point on the mesh $\left\{ \bar{t}_i \right\}_0^{2N}$, where $\bar{t}_i = \bar{t}_{i-1} + h/2$, $i = 1(1)2N$,

In the similar way, we can define $Z_{h/2}$ by replacing h by h/2 and N by 2N i.e.,

$$Z_{h/2} = \max_j \mid u_j^{h/2} - u_j^{h/4} \mid , \qquad j = 0,\ldots,2N-1$$

Now, the computed order of convergence is defined as:

$$\text{Order} = \frac{\left[\log Z_h - \log Z_{h/2} \right]}{\log 2}$$

We have taken $h = 1/5$ and $\lambda = 1/16$ for finding the computed order of convergence and results are shown in Table 3.4.

Table 3.1

Example 3.1

h		$\varepsilon = 10^{-3}$ $x_p = -1+2E-2$	$\varepsilon = 10^{-5}$ $x_p = -1+2E-4$
1/5	I_{max}	1.69395 E -6	1.69395 E -6
	I_{min}	2.53985 E -12	2.82659 E -11
	O_{max}	3.37505 E -9	3.37505 E -9
	O_{min}	0.0	0.0
1/10	I_{max}	1.04895 E -7	1.04895 E -7
	I_{min}	2.46469 E -14	5.26245 E -13
	O_{max}	3.73001 E -9	3.73001 E -9
	O_{min}	0.0	0.0
1/20	I_{max}	6.53482 E -9	6.53482 E -9
	I_{min}	1.54154 E -12	6.47732 E -11
	O_{max}	3.92125 E -9	3.92125 E -9
	O_{min}	0.0	0.0

Table 3.2

Example 3.2			
h		$\varepsilon = 10^{-3}$ $x_p = 0.4E-2$	$\varepsilon = 10^{-5}$ $x_p = 0.4E-4$
1/5	I_{max}	6.79166 E -4	5.18669 E -5
	I_{min}	1.55150 E -5	4.50113 E -7
	O_{max}	3.33692 E -4	4.53325 E -6
	O_{min}	0.0	0.0
1/10	I_{max}	3.89329 E -5	3.17298 E -6
	I_{min}	3.14849 E -6	5.34096 E -8
	O_{max}	3.32549 E -4	3.33874 E -6
	O_{min}	0.0	0.0
1/20	I_{max}	3.86764 E -5	1.36592 E -6
	I_{min}	2.31183 E -6	2.15286 E -9
	O_{max}	3.32249 E -4	3.33199 E -5
	O_{min}	0.0	0.0

Table 3.3

Example 3.3

h		$\varepsilon = 10^{-3}$ $x_p = 0.5E-2$	$\varepsilon = 10^{-5}$ $x_p = 0.5E-4$
1/5	I_{max}	5.65022 E -4	5.70341 E -5
	I_{min}	1.12327 E -4	1.32947 E -7
	O_{max}	4.97643 E -4	4.90557 E -6
	O_{min}	0.0	0.0
1/10	I_{max}	5.61162 E -4	5.95588 E -6
	I_{min}	4.97209 E -5	1.29740 E -7
	O_{max}	4.97732 E -4	4.99458 E -6
	O_{min}	0.0	0.0
1/20	I_{max}	5.59555 E -4	5.93288 E -6
	I_{min}	2.45255 E -5	4.01312 E -7
	O_{max}	4.97737 E -4	4.99925 E -6
	O_{min}	0.0	0.0

Table 3.4

	h	h/2	Z_h		Order
Example 3.1	1/5	1/10	1.49	E -14	
	1/10	1/20	1.20	E -15	3.63
Example 3.2	1/5	1/10	4.7050	E -4	
	1/10	1/20	2.5100	E -5	4.22
Example 3.2	1/5	1/10	7.0721	E -5	
	1/10	1/20	3.6132	E -6	4.29

CHAPTER IV

VARIABLE MESH DIFFERENCE SCHEME FOR LINEAR SINGULARLY PERTURBED BOUNDARY VALUE PROBLEMS

4.1 INTRODUCTION :

Singularly perturbed boundary value problems occur in many areas of engineering and applied mathematics. In many practical problems the coefficient of the second derivative is small as compared to the coefficient of the first derivative. Examples of these are heat transport problems with large Peclet numbers, Navier stokes flows with large Reynold numbers etcetra. Because of the presence of 'Boundary layers', difficulties are experienced in solving problems of above type using numerical methods with uniform mesh. In order to get good approximation a fine mesh is required in the boundary layer region.

In this chapter, we first consider the following linear two point boundary value problems without first derivative term

$$\varepsilon y'' = q(x)\, y + r(x) \tag{4.1}$$

$$y(a) = \alpha_0,\ y(b) = \alpha_1 \tag{4.2}$$

where $q(x) > 0$ and $r(x)$ are sufficiently smooth functions, α_0 and α_1 are constants, $0 < \varepsilon << 1$.

Under these assumptions, problem (4.1)-(4.2) has a unique solution and exhibits boundary layers at both ends of the interval (Ascher et. al 1988).

(i) $S(x)$ coincides with a polynomial of degree three on each $[x_i,x_{i+1}]$, $i=0(1)N-1$

(ii) $S(x) \in C^2[a,b]$

(iii) $S(x_i) = y(x_i)$, $i=0(1)N$

The cubic spline can be given in the form (Ahlberg et. al.1967).

$$S(x) = \frac{(x_{i+1}-x)^3}{6h_i} M_i + \frac{(x-x_i)^3}{6h_i} M_{i+1} + \left[y(x_i) - \frac{h_i^2 M_i}{6} \right]\left(\frac{x_{i+1}-x}{h_i} \right)$$

$$+ \left[y(x_{i+1}) - \frac{h_i^2 M_{i+1}}{6} \right]\left(\frac{x-x_i}{h_i} \right)$$

$$x_i \le x \le x_{i+1}, \quad i = 0(1)N-1 \tag{4.5}$$

where $M_i = S''(x_i)$, $i=0(1)N$

It's first derivative is given by

$$S'(x) = -M_i \frac{(x_{i+1}-x)^2}{2h_i} + M_{i+1}\frac{(x-x_i)^2}{2h_i} + \frac{y(x_{i+1}) - y(x_i)}{h_i} - \frac{M_{i+1}-M_i}{6} h_i$$

$$x_i \le x \le x_{i+1}, \quad i = 0(1)N-1 \tag{4.6}$$

and the second derivative is given by

$$S''(x) = M_i \frac{x_{i+1}-x}{h_i} + M_{i+1}\frac{x-x_i}{h_i} \tag{4.7}$$

For the one sided limit of the first derivative, from (4.6) we have

$$S'(x_i-) = \frac{h_{i-1}}{6} M_{i-1} + \frac{h_{i-1}}{3} M_i + \frac{y(x_i)-y(x_{i-1})}{h_{i-1}} \tag{4.8}$$

and $$S'(x_i+) = -\frac{h_i}{3} M_i - \frac{h_i}{6} M_{i+1} + \frac{y(x_{i+1}) - y(x_i)}{h_i} \tag{4.9}$$

We have derived a second order convergent variable difference scheme using cubic spline for the solution of (4.1)-(4.2). Then by taking second order approximations of first derivative terms, the method is generalised for the general linear singularly perturbed two point boundary value problem of the form:

$$\varepsilon\, y'' = p(x)y' + q(x)\, y + r(x) \tag{4.3}$$

$$y(a) = \alpha_0,\ y(b) = \alpha_1 \tag{4.4}$$

where $p(x)$, $q(x)$ and $r(x)$ are sufficiently smooth functions. Under these assumptions, problem (4.3)-(4.4) has a unique solution and exhibits boundary layers at one or both ends of the interval depending upon the properties of $p(x)$,(Ascher *et. al* 1988).

The main idea is to use the 'condition of continuity' of cubic spline as a discretization for (4.1) and (4.3). The use of cubic spline for the solution of regular linear two point boundary value problems without y' term was proposed by (Bickely 1968). Our scheme reduces to that of Bickley's for the corresponding regular problem on uniform mesh.

Also, piecewise $C^2[a,b]$ approximate solution for off-nodal points can be obtained without any further computation.

4.2 DIFFERENCE SCHEME FOR PROBLEMS WITHOUT Y' TERM :

$$\text{Let } x_0 = a,\quad x_i = a + \sum_{k=0}^{i-1} h_k,\ i=1(1)N,\ h_k = x_{k+1} - x_k,\ x_N = b$$

Given the values $y(x_0), y(x_1), \ldots, y(x_N)$ of a function $y(x)$ at the nodal points $x_0, x_1, \ldots, x_N$, there exists an interpolating cubic spline $S(x)$ with the following properties :

(i) $S(x)$ coincides with a polynomial of degree three on each $[x_i, x_{i+1}]$, $i=0(1)N-1$

(ii) $S(x) \in C^2[a,b]$

(iii) $S(x_i) = y(x_i)$, $i=0(1)N$

The cubic spline can be given in the form (Ahlberg et. al.1967).

$$S(x) = \frac{(x_{i+1}-x)^3}{6h_i} M_i + \frac{(x-x_i)^3}{6h_i} M_{i+1} + \left[y(x_i) - \frac{h_i^2 M_i}{6} \right]\left(\frac{x_{i+1}-x}{h_i} \right)$$
$$+ \left[y(x_{i+1}) - \frac{h_i^2 M_{i+1}}{6} \right]\left(\frac{x-x_i}{h_i} \right)$$
$$x_i \le x \le x_{i+1}, \quad i = 0(1)N-1 \qquad (4.5)$$

where $M_i = S''(x_i)$, $i=0(1)N$

It's first derivative is given by

$$S'(x) = -M_i \frac{(x_{i+1}-x)^2}{2h_i} + M_{i+1} \frac{(x-x_i)^2}{2h_i} + \frac{y(x_{i+1}) - y(x_i)}{h_i} - \frac{M_{i+1} - M_i}{6} h_i$$

$$x_i \le x \le x_{i+1}, \quad i = 0(1)N-1 \qquad (4.6)$$

and the second derivative is given by

$$S''(x) = M_i \frac{x_{i+1}-x}{h_i} + M_{i+1} \frac{x-x_i}{h_i} \qquad (4.7)$$

For the one sided limit of the first derivative, from (4.6) we have

$$S'(x_i-) = \frac{h_{i-1}}{6} M_{i-1} + \frac{h_{i-1}}{3} M_i + \frac{y(x_i)-y(x_{i-1})}{h_{i-1}} \qquad (4.8)$$

and $$S'(x_i+) = -\frac{h_i}{3} M_i - \frac{h_i}{6} M_{i+1} + \frac{y(x_{i+1}) - y(x_i)}{h_i} \qquad (4.9)$$

By (4.5) and (4.7), the functions $S(x)$ and $S''(x)$ are continuous on $[a,b]$ and for $S'(x)$ to be continuous at the interior node x_i, we have from (4.8)-(4.9), the following condition known as the 'condition of continuity'

$$\frac{h_{i-1}}{6} M_{i-1} + \frac{h_i + h_{i-1}}{3} M_i + \frac{h_i}{6} M_{i+1} = \frac{y(x_{i+1})-y(x_i)}{h_i} - \frac{y(x_i)-y(x_{i-1})}{h_{i-1}}$$

$$i=1(1)N-1 \qquad (4.10)$$

The 'condition of continuity' ensures the continuity of the first derivatives of the spline $S(x)$ at the interior nodes.

Multiplying and dividing the terms containing h_{i-1} by h_i and putting $\sigma_i = \frac{h_i}{h_{i-1}}$ we get

$$\frac{h_i}{6\sigma_i} M_{i-1} + \frac{h_i}{3}(1 + \frac{1}{\sigma_i})M_i + \frac{h_i}{6} M_{i+1} = \frac{y(x_{i+1}) - y(x_i)}{h_i}$$

$$- \frac{\sigma_i[(y(x_i)- y(x_{i-1})]}{h_i} \qquad (4.11)$$

Substituting the following in Eq.(4.11)

$$\varepsilon M_j = q(x_j)y(x_j) + r(x_j), \; j = i,i\pm1$$

we have

$$\frac{h_i}{6\sigma_i} [q_{i-1}y(x_{i-1}) + r_{i-1})] + \frac{h_i}{3} (1 + \frac{1}{\sigma_i})[(q_i y(x_i) + r_i)]$$

$$+ \frac{h_i}{6} [q_{i+1}y(x_{i+1}) + r_{i+1}] = \frac{\varepsilon[y(x_{i+1})- y(x_i)]}{h_i} - \frac{\varepsilon\sigma_i[y(x_i)-y(x_{i-1})]}{h_i}$$

$$(4.12)$$

Neglecting the truncation error, we get the following tridiagonal system for approximating $y_1, y_2, \ldots y_{N-1}$ at the nodal points $x_1, x_2, \ldots, x_{N-1}$

$$\left[- \varepsilon\,\sigma_i + \frac{h_i^2}{6\sigma_i}\, q_{i-1} \right] y_{i-1} + \left[\varepsilon(1+\sigma_i) + \frac{h_i^2(1+\sigma_i)}{3\sigma_i}\, q_i \right] y_i$$

$$+ \left[-\varepsilon + \frac{h_i^2}{6}\, q_{i+1} \right] y_{i+1} = - h_i^2 \left[\frac{r_{i-1}}{6\sigma_i} + \frac{(1+\sigma_i)}{3\sigma_i}\, r_i + \frac{r_{i+1}}{6} \right]$$

$$i=1(1)N-1 \qquad (4.13)$$

with $y_0 = \alpha_0$, $y_N = \alpha_1$

where $q_i = q(x_i)$, $r_i = r(x_i)$, $i = 0(1)N$, $\sigma_i = h_i/h_{i-1}$, $i=1(1)N-1$

__Remark 4.1__

For $\varepsilon = 1$(regular problem) and $\sigma_i \equiv 1$ (uniform mesh), we have

$$\left[-1 + \frac{h^2}{6} q_{i-1}\right] y_{i-1} + \left[2 + \frac{4h^2}{6}\, q_i \right] y_i + \left[-1 + \frac{h^2}{6}\, q_{i+1} \right] y_{i+1}$$

$$= - \frac{h^2}{6} \left[r_{i-1} + 4\, r_i + r_{i+1} \right], \quad i=1(1)N-1 \qquad (4.14)$$

$y_0 = \alpha_0$ and $y_1 = \alpha_1$

which is Bickley's scheme for the regular problem

$$y'' = q(x)\; y + r(x) \qquad (4.15)$$

$$y(a) = \alpha_0, \; y(b) = \alpha_1 \qquad (4.16)$$

4.3 DIFFERENCE SCHEME FOR PROBLEMS WITH Y′ TERM :

In this section, we have generalized the scheme derived in previous section by taking some second order approximation of the first derivative for the problem (4.3)-(4.4).

Taking the usual Taylor series expansion for y around x_i and neglecting the terms containing third and higher order terms, we get the following approximations for y_{i+1} and y_{i-1}:

$$y_{i+1} \simeq y_i + h_i y_i' + \frac{h_i^2}{2} y_i'' \tag{4.17}$$

$$y_{i-1} \simeq y_i - h_{i-1} y_i' + \frac{h_{i-1}^2}{2} y_i'' \tag{4.18}$$

Multiplying (4.18) by σ_i^2 and subtracting it from (4.17), we get the following approximation for y_i'.

$$y_i' \simeq \frac{1}{h_i(1+\sigma_i)} \left[y_{i+1} + (\sigma_i^2 - 1) y_i - \sigma_i^2 y_{i-1} \right] \tag{4.19}$$

Multiplying (4.18) by σ_i and adding it to (4.17), we get the following approximation for y_i''

$$y_i'' \simeq \frac{2\sigma_i}{h_i(1+\sigma_i)^2} \left[y_{i+1} - (1 + \sigma_i) y_i + \sigma_i y_{i-1} \right] \tag{4.20}$$

Also we have

$$y_{i+1}' \simeq y_i' + h_i\, y_i'' \tag{4.21}$$

$$y_{i-1}' \simeq y_i' - h_{i-1}\, y_i'' \tag{4.22}$$

Using the expressions for y_i' and y_i'' from (4.19) and (4.20) respectively and putting them in (4.21) we get the following approximation for y_{i+1}',

$$y'_{i+1} \simeq \frac{1}{h_i(1+\sigma_i)}\left[(2\sigma_i+1)y_{i+1} - (\sigma_i+1)^2 y_i + \sigma_i^2 y_{i-1}\right]$$

$$i=1(1)N-1 \qquad (4.23)$$

Similarly, using the expressions for y'_i and y''_i from (4.19) and (4.20) respectively and putting them in (4.22) we get the following approximation for y'_{i-1},

$$y'_{i-1} \simeq \frac{1}{h_i(1+\sigma_i)}\left[-y_{i+1} + (\sigma_i + 1)^2\, y_i - \sigma_i(\sigma_i + 2)\, y_{i-1}\right]$$

$$i=1(1)N-1 \qquad (4.24)$$

Substituting the following in Eq.(4.10)

$$\varepsilon\, M_j = p(x_j)y'_j + q(x_j)y_j + r(x_j),\ j = i, i\pm 1$$

and using Eqs. (4.19), (4.23) and (4.24) for the first order derivatives, we get the following system which gives the approximations for $y_1, y_2, \ldots, y_{N-1}$.

$$\left[-\varepsilon\,\sigma_i + \frac{h_i^2}{6\sigma_i}q_{i-1} - \frac{h_i(\sigma_i+2)}{6(1+\sigma_i)}p_{i-1} - \frac{h_i\sigma_i}{3}p_i + \frac{h_i\sigma_i^2}{6(1+\sigma_i)}p_{i+1}\right]y_{i-1}$$

$$+\left[\varepsilon(1+\sigma_i) + \frac{h_i^2(1+\sigma_i)}{3\sigma_i}q_i + \frac{h_i(1+\sigma_i)}{6\sigma_i}p_{i-1} - \frac{h_i(\sigma_i^2-1)}{3\sigma_i}p_i - \frac{h_i(\sigma_i+1)}{6}p_{i+1}\right]y_i$$

$$+\left[-\varepsilon + \frac{h_i^2}{6}q_{i+1} - \frac{h_i}{6(\sigma_i+1)\sigma_i}p_{i-1} + \frac{h_i}{3\sigma_i}p_i + \frac{h_i(2\sigma_i+1)}{6(\sigma_i+1)}p_{i+1}\right]y_{i+1}$$

$$= -h_i^2\left[\frac{r_{i-1}}{6\sigma_i} + \frac{(1+\sigma_i)}{3\sigma_i}r_i + \frac{r_{i+1}}{6}\right] \qquad i=1(1)N-1 \qquad (4.25)$$

with $y_0 = \alpha_0,\ y_N = \alpha_1$

where $p_i = p(x_i)$, $f_i = f(x_i)$, $i = 0(1)N$, $\sigma_i = h_i/h_{i-1}$, $i=1(1)N$

For selecting the mesh points, we follow the procedure given by Jain et. al(1984).

4.4 MESH SELECTION PROCEDURE :

Let N be the number of mesh points in the interval [a,b] and $\sigma_i (=h_i/h_{i-1})$ be the mesh ratio (as defined earlier). For simplicity we take $\sigma_1 = \sigma$ (a constant) $\forall$ i.

We have

$$\begin{aligned} b - a &= x_N - x_0 \\ &= (x_N - x_{N-1}) + (x_{N-1} - x_{N-2}) + \ldots + (x_2 - x_1) + (x_1 - x_0) \\ &= \sigma^{N-1} h_0 + \sigma^{N-2} h_0 + \ldots + \sigma h_0 + h_0 \\ &= (\sigma^{N-1} + \sigma^{N-2} + \ldots + \sigma + 1) h_0 \end{aligned}$$

which implies

$$h_0 = \begin{cases} \dfrac{(b-a)(\sigma-1)}{(\sigma^N - 1)} & \text{if } \sigma > 1 \\[2ex] \dfrac{(b-a)(1-\sigma)}{(1-\sigma^N)} & \text{if } \sigma < 1 \end{cases} \tag{4.26}$$

Therefore, given the values of N and σ, we can choose h_0 from (4.26) and subsequent h_i's can be obtained as $h_i = \sigma h_{i-1}$, $i=1(1)N-1$

CASE I

If the boundary layer occurs at the left end then we choose $\sigma > 1$. This ensures more mesh points near the left end of the interval.

CASE II

If the boundary layer occurs at the right end then we choose $\sigma < 1$. This ensures more mesh points near the right end of the interval.

CASE III

If there are boundary layers at both ends then we consider first half of the interval $[a,(a+b)/2]$ with $\sigma > 1$, then we take its mirror image in other half interval $[(a+b)/2,b]$. This ensures more mesh points at both ends.

CASE IV

If there is boundary layer at the center then we consider first half of the interval $[a,(a+b)/2]$ with $\sigma < 1$, then we take its mirror image in other half interval $[(a+b)/2,b]$. This ensures more mesh points at the center.

4.5 ERROR ANALYSIS :

Taking the usual Taylor series expansion for y around x_i, we get the following expressions for y_{i+1} and y_{i-1}:

$$y_{i+1} = y_i + h_i y_i' + \frac{h_i^2}{2} y_i'' + \frac{h_i^3}{6} y'''(\xi_1^{(1)}) \tag{4.27}$$

$$y_{i-1} = y_i - h_{i-1} y_i' + \frac{h_{i-1}^2}{2} y_i'' - \frac{h_{i-1}^3}{6} y'''(\xi_2^{(1)}) \tag{4.28}$$

$$x_{i-1} < \xi_1^{(1)},\ \xi_2^{(1)} < x_{i+1}$$

Multiplying (4.28) by σ_i^2, and subtracting it from (4.27), we get the following expression for y_i'

$$y_i' = y'(x_i) + \frac{h_i^2}{6(1+\sigma_i)} \left[y'''(\xi_1^{(i)}) + \frac{1}{\sigma_i} y'''(\xi_2^{(i)}) \right] \tag{4.29}$$

Multiplying (4.28) by σ_i, and subtracting it from (4.27), we get the following expression for y_i''

$$y_i'' = y''(x_i) + \frac{h_i}{6} \left[y'''(\xi_1^{(i)}) - \frac{1}{\sigma_i^2} y'''(\xi_2^{(i)}) \right] \tag{4.30}$$

Also, we have

$$y_{i+1}' = y_i' + h_i y_i'' + \frac{h_i^2}{2} y'''(\xi_3^{(i)}) \tag{4.31}$$

$$x_{i-1} < \xi_3^{(i)} < x_{i+1}$$

Putting the values of y_i' and y_i'' from (4.29) and (4.30) in (4.31), we have

$$y_{i+1}' = y'(x_{i+1}) + \frac{h_i^2}{2} y'''(\xi_3^{(i)}) + \frac{h_i^2}{6}\left[\left(\frac{2+\sigma_i}{1+\sigma_i} \right) y'''(\xi_1^{(i)}) + \left(\frac{\sigma_i - 1}{\sigma_i^2(\sigma_i+1)}\right) y'''(\xi_2^{(i)}) \right] \tag{4.32}$$

Similarly, we have

$$y_{i-1}' = y_i' - h_{i-1} y_i'' + \frac{h_{i-1}^2}{2} y'''(\xi_4^{(i)}) \tag{4.33}$$

$$x_{i-1} < \xi_4^{(i)} < x_{i+1}$$

Putting the values of y_i' and y_i'' from (4.29) and (4.30) in (4.33), we have

$$y_{i-1}' = y'(x_{i-1}) + \frac{h_i^2}{2\sigma_i^2} y'''(\xi_4^{(1)}) + \frac{h_i^2}{6}\left[\left(-\frac{1}{\sigma_i(1+\sigma_i)}\right)y'''(\xi_1^{(1)}) + \left(\frac{2\sigma_i+1}{\sigma_i^2(\sigma_i+1)}\right)y'''(\xi_2^{(1)})\right] \qquad (4.34)$$

Expressions (4.29),(4.32) and (4.34) show that y_i', y_{i+1}' and y_{i-1}' are second order approximations to $y'(x_i)$, $y'(x_{i+1})$ and $y'(x_{i-1})$ respectively.

Putting the tridiagonal system (4.25) in the matrix form, we have

$$MY = R \qquad (4.35)$$

where $M = (m_{i,j})$, $1 \le i,j \le N-1$ is a tridiagonal matrix with,

$$\begin{aligned} m_{i,i+1} &= \text{coefficient of } y_{i+1} \text{ in (4.25)} \quad i=1(1)N-2 \\ m_{i,i} &= \text{coefficient of } y_i \text{ in (4.25)} \quad i=1(1)N-1 \\ m_{i,i-1} &= \text{coefficient of } y_{i-1} \text{ in (4.25)} \quad i=2(1)N-2 \end{aligned}$$

and $R = (\bar{r}_i)$, $1 \le i \le N-1$ is a column vector with

$$\bar{r}_i = -\frac{h_i^2}{6}\left[\frac{1}{\sigma_i} r_{i-1} + \frac{2(1+\sigma_i)}{\sigma_i} r_i + r_{i+1}\right], \qquad i=1(1)N-1$$

and $Y = (y_1, y_2, \ldots, y_{N-1})^{tr}$

Also we have

$$MY_A - T(h) = R \qquad (4.36)$$

where $Y_A = (y(x_1), y(x_2), \ldots, y(x_{N-1}))^{tr}$ denotes the actual solution

and $T(h) = \left[T(h_1), T(h_2), \ldots, T(h_{N-1}) \right]^{tr}$

where $T(h_i) = \dfrac{\varepsilon\sigma_i^3}{24} h_i^4 y^{(iv)}(\xi_i), \quad x_{i-1} < \xi_i < x_{i+1}$ (4.37)

is the truncation error

From (4.35) and (4.36), we have

$$M(Y - Y_A) = T(h) \tag{4.38}$$

Thus, the error equation is

$$ME = T(h) \tag{4.39}$$

where $E = Y - Y_A$

Also, it can be seen that for h_i satisfying following conditions for different cases, the tridiagonal matrix M is irreducibly diagonally dominant.

Case I $\qquad p(x) > 0, \quad q(x) < 0$

For $m_{i,i-1} < 0$, we must have

$$A_1 + h_i^2 A_2 - h_1 A_3 - h_i A_4 + h_i A_5 < 0 \tag{4.40}$$

where $A_1 = \varepsilon\, \sigma_i$

$A_2 = \dfrac{1}{6\sigma_i} q_{i-1}$

$A_3 = \dfrac{\sigma_i + 2}{6(1+\sigma_i)} p_{i-1}$

$A_4 = \dfrac{\sigma_i}{3} p_i$

$A_5 = \dfrac{\sigma_i^2}{6(1+\sigma_i)} p_{i+1}$

Since $A_1 > 0$, it is enough to have

$$h_i^2 A_2 - h_i A_3 - h_i A_4 + h_i A_5 < 0 \tag{4.41}$$

Since $A_2 < 0$, this implies

$$\frac{1}{A_2}(A_3 + A_4 - A_5) < h_i \tag{4.42}$$

Let $H_1 = \frac{1}{A_2} (A_3 + A_4 - A_5)$

$$\text{Then } H_1 = \frac{\sigma_i}{(1+\sigma_i) q_{i-1}} \left[(\sigma_i+2) p_{i-1} + 2\sigma_i (1+\sigma_i) - \sigma_i^2 p_{i+1} \right] \tag{4.43}$$

Therefore, for $m_{i,i-1} < 0$, we must have

$$H_1 < h_i \tag{4.44}$$

Now for $m_{i,i} > 0$, we must have

$$B_1 + h_i^2 B_2 + h_i B_3 - h_i B_4 - h_i B_5 > 0 \tag{4.45}$$

where

$$B_1 = \varepsilon (1+\sigma_i)$$

$$B_2 = \frac{1+\sigma_i}{3\sigma_i} q_i$$

$$B_3 = \frac{1+\sigma_i}{6\sigma_i} p_{i-1}$$

$$B_4 = \frac{\sigma_i^2-1}{3\sigma_i} p_i$$

$$B_5 = \frac{1+\sigma_i}{6} p_{i+1}$$

Since $B_1 > 0$, it is enough to have

$$h_i^2 B_2 + h_i B_3 - h_i B_4 - h_i B_5 > 0 \tag{4.46}$$

Since $B_2 < 0$, this implies

$$h_i < -\frac{1}{B_2}\left(B_3 - B_4 - B_5 \right) \tag{4.47}$$

Let $H_2 = -\frac{1}{B_2}\left(B_3 - B_4 - B_5 \right)$

Then $$H_2 = \frac{1}{-2q_i}\left[p_{i-1} + 2(\sigma_i - 1)\, p_i - \sigma_i p_{i+1}\right] \tag{4.48}$$

Therefore, for $m_{i,i} > 0$, we must have

$$h_i < H_2 \tag{4.49}$$

For $m_{1,i+1} < 0$, we must have

$$- C_1 + h_i^2 C_2 - h_i C_3 + h_i C_4 + h_i C_5 < 0 \tag{4.50}$$

where $C_1 = \varepsilon$

$C_2 = \frac{1}{6} \quad q_{i+1}$

$C_3 = \frac{1}{6(1+\sigma_i)\sigma_i}\, p_{i-1}$

$C_4 = \frac{1}{3\sigma_1}\, p_i$

$C_5 = \frac{2\sigma_i + 1}{6(1+\sigma_i)} p_{i+1}$

Since $C_1 > 0$, it is enough to have

$$h_i^2 C_2 - h_i C_3 + h_i C_4 + h_i C_5 < 0 \tag{4.51}$$

Since $C_2 < 0$, this implies

$$\frac{1}{C_2}\left(C_3 - C_4 - C_5 \right) < h_i \tag{4.52}$$

Let $H_3 = \frac{1}{C_2} (C_3 - C_4 - C_5)$

Then $$H_3 = \frac{-1}{\sigma_i(1+\sigma_i)q_{i+1}}\left[p_{i-1} + 2(1+\sigma_i)\, p_i + 2(1+\sigma_i)\, p_{i+1}\right] \tag{4.53}$$

Therefore, for $m_{1,i+1} < 0$, we must have

$$H_3 < h_i \tag{4.54}$$

Combining the conditions (4.44),(4.49) and (4.54), we have

$$\max(H_1,H_3) < h_i < H_2 \tag{4.55}$$

Similarly, for other cases, we have the following conditions

<u>Case II</u> $\quad p(x) > 0, \quad q(x) > 0$

$$H_1 < h_i < H_2 \tag{4.56}$$

Where $$H_1 = -\frac{1}{q_{1-1}}\left[p_{i-1} + (\sigma_i-1)\, p_i - 2\sigma_1\, p_{i+1}\right]$$

and $$H_2 = \frac{\sigma_i}{q_{i+1}}\left[\frac{\sigma_i+2}{\sigma_i+1}\, p_{i-1} + 2\sigma_i p_i + \frac{\sigma_i^2}{1+\sigma_i} p_{i+1}\right]$$

<u>Case III</u> $\quad p(x) < 0, \quad q(x) < 0$

$$\max(H_1,H_2) < h_i < H_3 \tag{4.57}$$

where $$H_1 = \frac{6\sigma_i}{q_{i-1}}\left[\frac{\sigma_i+2}{\sigma_i(\sigma_i+1)}\, p_{1-1} - \frac{\sigma_i}{3}\, p_1 - \frac{\sigma_i^2}{6(1+\sigma_i)}\, p_{i+1}\right]$$

$$H_2 = \frac{1}{q_{i-1}}\left[\frac{1}{\sigma_i(\sigma_i+1)} p_{i-1} - \frac{2}{\sigma_i}\, p_i - \frac{2\sigma_i+1}{\sigma_i+1}\, p_{1+1}\right]$$

and $$H_3 = \frac{-3\sigma_i}{(1+\sigma_i)q_1}\left[\frac{\sigma_i^2-1}{3\sigma_i}\, p_i - \frac{\sigma_i+1}{6}\, p_{1+1} + \frac{\sigma_i+1}{6\sigma}\, p_{i-1}\right]$$

<u>Case IV</u> $p(x) < 0, \quad q(x) > 0$

$$H_1 < h_1 < \min (H_2, H_3) \tag{4.58}$$

Where $H_1 = \frac{1}{2q_i}\left[\sigma_i p_{i+1} + (1 - \sigma_i)p_i - p_{i-1}\right]$

$$H_2 = \frac{\sigma_i}{q_{i-1}}\left[\frac{\sigma_i+2}{\sigma_i+1} p_{i-1} + 2\sigma_i p_i - \frac{\sigma_i^2}{1+\sigma_i} p_{i+1} \right]$$

and $H_3 = \frac{1}{q_{i+1}}\left[\frac{1}{\sigma_i(\sigma_i+1)} p_{i-1} - 2p_i - \frac{2\sigma_i+1}{\sigma_i+1} p_{i+1}\right]$

<u>Case V</u> $p(x) \equiv 0, \quad q(x) > 0$

$$h_i < \min (H_1, H_2) \tag{4.59}$$

where $H_1 = \sigma_i\sqrt{\frac{6\varepsilon}{q_{i-1}}}$ and $H_2 = \sqrt{\frac{6\varepsilon}{q_{i-1}}}$

<u>Case VI</u> $p(x) < 0, \quad q(x) \equiv 0$

$$h_i < \min (H_1, H_2, H_3) \tag{4.60}$$

where $H_1 = \frac{\varepsilon\sigma_1}{\sigma_1(1+\sigma_i)} p_{i+1}$

$$H_2 = \frac{6\sigma_i\varepsilon(\sigma_i+1)}{2(\sigma_i+1)p_i+\sigma_i(2\sigma_i+1)p_{i+1}-p_{i-1}}$$

and $H_3 = \frac{6\sigma_i\varepsilon}{\sigma_i p_{i+1}-2(\sigma_i - 1)p_i-p_{i-1}}$

<u>Case VII</u> $p(x) > 0, \quad q(x) \equiv 0$

$$h_i < \min (H_1, H_2, H_3) \tag{4.61}$$

<u>Case IV</u> $\quad p(x) < 0, \quad q(x) > 0$

$$H_1 < h_i < \min (H_2, H_3) \tag{4.58}$$

Where $H_1 = \frac{1}{2q_i}\left[\sigma_i p_{i+1} + (1 - \sigma_i)p_i - p_{i-1}\right]$

$$H_2 = \frac{\sigma_i}{q_{i-1}}\left[\frac{\sigma_i+2}{\sigma_i+1}\, p_{i-1} + 2\sigma_i p_i - \frac{\sigma_i^2}{1+\sigma_i} p_{i+1} \right]$$

and $H_3 = \frac{1}{q_{i+1}}\left[\frac{1}{\sigma_i(\sigma_i+1)}\, p_{i-1} - 2p_i - \frac{2\sigma_i+1}{\sigma_i+1}\, p_{i+1}\right]$

<u>Case V</u> $\quad p(x) \equiv 0, \quad q(x) > 0$

$$h_i < \min (H_1, H_2) \tag{4.59}$$

where $H_1 = \sigma_i\sqrt{\frac{6\varepsilon}{q_{i-1}}}$ and $H_2 = \sqrt{\frac{6\varepsilon}{q_{i-1}}}$

<u>Case VI</u> $\quad p(x) < 0, \quad q(x) \equiv 0$

$$h_i < \min (H_1, H_2, H_3) \tag{4.60}$$

where $H_1 = \frac{\varepsilon\sigma_i}{\sigma_1(1+\sigma_i)}\, p_{i+1}$

$$H_2 = \frac{6\sigma_i\varepsilon(\sigma_i+1)}{2(\sigma_i+1)p_i+\sigma_i(2\sigma_i+1)p_{i+1}-p_{i-1}}$$

and $H_3 = \frac{6\sigma_i\varepsilon}{\sigma_i p_{i+1}-2(\sigma_i - 1)p_i - p_{i-1}}$

<u>Case VII</u> $\quad p(x) > 0, \quad q(x) \equiv 0$

$$h_i < \min (H_1, H_2, H_3) \tag{4.61}$$

Case IV $\quad p(x) < 0, \quad q(x) > 0$

$$H_1 < h_i < \min (H_2, H_3) \tag{4.58}$$

Where $H_1 = \frac{1}{2q_i}\left[\sigma_i p_{i+1} + (1 - \sigma_i)p_i - p_{i-1}\right]$

$$H_2 = \frac{\sigma_i}{q_{i-1}}\left[\frac{\sigma_i+2}{\sigma_i+1} p_{i-1} + 2\sigma_i p_i - \frac{\sigma_i^2}{1+\sigma_i} p_{i+1} \right]$$

and $H_3 = \frac{1}{q_{i+1}}\left[\frac{1}{\sigma_i(\sigma_i+1)} p_{i-1} - 2p_i - \frac{2\sigma_i+1}{\sigma_i+1} p_{i+1}\right]$

Case V $\quad p(x) \equiv 0, \quad q(x) > 0$

$$h_i < \min (H_1, H_2) \tag{4.59}$$

where $H_1 = \sigma_i\sqrt{\frac{6\varepsilon}{q_{i-1}}}$ and $H_2 = \sqrt{\frac{6\varepsilon}{q_{i-1}}}$

Case VI $\quad p(x) < 0, \quad q(x) \equiv 0$

$$h_i < \min (H_1, H_2, H_3) \tag{4.60}$$

where $H_1 = \frac{\varepsilon\sigma_i}{\sigma_i(1+\sigma_i)} p_{i+1}$

$$H_2 = \frac{6\sigma_i\varepsilon(\sigma_i+1)}{2(\sigma_i+1)p_i+\sigma_i(2\sigma_1+1)p_{i+1}-p_{i-1}}$$

and $H_3 = \frac{6\sigma_i\varepsilon}{\sigma_i p_{i+1}-2(\sigma_i-1)p_i-p_{i-1}}$

Case VII $\quad p(x) > 0, \quad q(x) \equiv 0$

$$h_i < \min (H_1, H_2, H_3) \tag{4.61}$$

$$e_j = \sum_{i=1}^{N-1} m_{j,i}^{-1} T(h) \qquad j=1(1)N-1 \qquad (4.67)$$

and therefore

$$|e_j| \le \frac{Ch^4}{h_{i_0}^2 |D_{i_0}|} \qquad j=1(1)N-1 \qquad (4.68)$$

where C is constant independent of $h = \max\limits_{1\le i\le N-1} h_i$

Therefore,

$$\| E \| = O\,(h^2) \qquad h = \max_{1\le i\le N-1} h_i \qquad (4.69)$$

4.6 NUMERICAL EXAMPLES :

Example 4.1 (Doolan et. al. 1980)

$$\varepsilon\, y'' = y + \cos^2 \pi x + 2\pi^2 \cos 2\pi x$$

$$y(0) = y(1) = 0$$

The exact solution is given by

$$y(x) = \frac{\exp(-(1+x)/\sqrt{\varepsilon}) + \exp(-x/\sqrt{\varepsilon})}{1 + \exp(-1/\sqrt{\varepsilon})} - \cos^2 \pi x$$

Since $p(x) \equiv 0$ and $q(x)\equiv 1 > 0$, the boundary layer exists at both ends, so we choose mesh according to case III of the section 4.4.

Example 4.2 (Jain et. al. 1984)

$$\varepsilon y'' = y'$$

$$y(0) = 1, \; y(1) = 0$$

The exact solution is given by

$$y(x) = \frac{1-\exp(-(1-x)/\varepsilon)}{1-\exp(-1/\varepsilon)}$$

Since $p(x) \equiv 1 > 0$, the boundary layer exists near the right end, so we choose mesh according to case II of section 4.4.

Example 4.3 (Reinhardt 1980)

$$\varepsilon y'' = - y' + 1 + 2x$$

$$y(0)=0 \ , \ y(1)=1$$

The exact solution is given by

$$y(x) = x(x+1-2\varepsilon) + (2\varepsilon-1) \frac{1-\exp(-x/\varepsilon)}{1-\exp(-1/\varepsilon)}$$

Since $p(x) \equiv -1 < 0$, the boundary layer exists near left end, so we choose mesh according to case I of section 4.4.

4.7 DISCUSSION :

The second order variable mesh difference scheme, obtained from the 'condition of continuity' of the cubic spline, can be applied to any linear singularly perturbed boundary value problem, provided that there is some prior information about the location of boundary layer, so that mesh can be chosen accordingly.

We have implemented our method on three examples. Maximum error at the nodal points, $\max_i | y(x_i) - y_i |$, are tabulated in Tables 4.1-4.3 for different values of the parameters ε, N and σ(for simplicity we have taken $\sigma_i = \sigma \ \forall \ i$).

It is observed that errors are less for variable mesh ($\sigma \neq 1$) compared to that of uniform mesh ($\sigma = 1$), which shows the need of the variable mesh for the problem with boundary layer region where the solution changes rapidly. The uniform mesh gives good results when mesh is taken very fine through out the interval, which is quite demanding on the computer time. Also solution at any off-nodal point can be obtained from the cubic spline expression.

Table 4.1

Max. absolute error (Example 4.1)

	N	σ = 1.00	1.05	1.15
$\varepsilon = 10^{-5}$	60	1.87947 E +1	8.29368 E -2	3.81489 E -3
	120	9.54587 E -2	3.63303 E -3	6.63985 E -4
$\varepsilon = 10^{-8}$	100	2.67670 E -1	2.56806 E -1	9.78686 E -3
	200	2.66837 E -1	5.97463 E -2	6.43276 E -4
$\varepsilon = 10^{-10}$	150	2.67942 E -1	2.60797 E -1	2.02680 E -3
	300	2.67924 E -1	4.69272 E -2	6.41468 E -4
$\varepsilon = 10^{-12}$	200	2.67949 E -1	2.61329 E -1	9.44750 E -4
	400	2.67948 E -1	3.66378 E -2	6.40522 E -4

Table 4.2

Max. absolute error (Example 4.2)

	N	σ = 1.00	0.95	0.85
$\varepsilon = 10^{-5}$	60	6.97322	2.37996	5.14372 E -2
	120	1.91001	8.08277 E -1	3.20425 E -3
$\varepsilon = 10^{-8}$	120	1.73610 E +3	1.99962	4.14263 E -3
	240	4.34024 E +2	1.98215	3.19924 E -3
$\varepsilon = 10^{-10}$	250	4.00000 E +4	1.96678	3.20501 E -3
	500	1.00000 E +4	3.38173 E -4	3.20504 E -3
$\varepsilon = 10^{-12}$	300	2.77777 E +6	1.96644	3.22245 E -3
	600	6.94444 E +5	7.33741 E -5	3.22245 E -3

Table 4.3

Max. absolute error (Example 4.3)

	N	σ = 1.00	σ = 1.05	σ = 1.15
$\varepsilon = 10^{-5}$	80	7.84081	2.35713	6.09777 E -3
	160	2.11087	1.48059	2.36813 E -3
$\varepsilon = 10^{-8}$	120	3.47221 E +3	2.01874	2.83734 E -2
	240	8.68051 E +2	1.97420 E +2	2.36227 E -3
$\varepsilon = 10^{-10}$	150	2.22222 E +5	1.98307	6.39184 E -2
	300	5.55555 E +4	1.96816	2.36184 E -3
$\varepsilon = 10^{-12}$	200	1.25000 E +7	1.97073	4.00547 E -3
	400	3.12550 E +6	1.96800	2.36315 E -3

CHAPTER V

Third Order Variable Mesh Methods For Semi-Linear Singularly Perturbed Boundary Value Problems

5.1 INTRODUCTION :

Consider the following class of semi-linear singularly perturbed boundary value problems

$$\varepsilon y'' = f(x,y) \qquad a < x < b \ , \quad 0 < \varepsilon << 1 \tag{5.1}$$

$$y(a) = \alpha, \quad y(b) = \beta \tag{5.2}$$

In many fields of application, including geophysical fluid dynamics (Carrier,1970), these types of problems arise which exhibit boundary layer of small thickness at both ends. Difference methods with uniform mesh are not suitable to these type of problems as a fine mesh is required in boundary layer region and comparatively much coarser mesh elsewhere.

It is known that, under suitable assumptions, one expects the solution $y(x)$ of (5.1)-(5.2) to behave qualitatively as the solution of the following problem (Howes, 1980).

$$\varepsilon y'' = p(x)y + q(x) \qquad p(x) \geq p > 0 \tag{5.3}$$

$$y(a) = \alpha \ , \quad y(b) = \beta \tag{5.4}$$

where $p(x)$ and $q(x)$ are sufficiently smooth functions.

In this chapter, a two parameter family of methods based on cubic spline approximation, which gives third order approximation to the solution of (5.3)-(5.4), is presented. The methods are then used for solving (5.1)-(5.2) by quasi-linearization technique. For approximating solution at the nodal points, we have given a family of third order variable mesh methods. In addition to the parameter σ_i, one more parameter λ is introduced which is exploited to yield higher order approximations of the solution at off-nodal points. These third order variable mesh methods reduce to a one parameter family of fourth order methods when the mesh ratio is taken to be equal to unity i.e for the uniform mesh. In addition, when the parameter λ is equal to unity then we have well known Numerov's method for the corresponding regular problem. Also, the restriction on mesh size for convergence can be relaxed by suitably choosing the value of the parameter λ.

Earlier, Jain et al. (1984) have given a one parameter family of third order methods for the solution of $y'' = f(x,y,y')$ where f contains the small parameter ε implicitly. These methods reduces to one particular method namely Numerov's method for uniform mesh. Their methods become particular cases of our methods when the parameter $\lambda = 1$ for the problem of the form (5.3)-(5.4).

In order to get family of higher order methods instead of one particular second order method, as derived in the previous chapter, we proceed in the following way.

5.2 DERIVATION OF THIRD ORDER METHODS :

Let $x_o = a, \quad x_i = a + \sum_{k=o}^{i-1} h_k, \quad h_k = x_{k+1} - x_k, \quad x_N = b.$

Consider the Ahlberg's cubic spline, considered in the previous chapter.

$$S(x) = \frac{(x_{i+1} - x)^3}{6h_i} M_i + \frac{(x - x_i)^3}{6h_i} M_{i+1} + \left[y(x_i) - \frac{h_i^2 M_i}{6} \right]\left(\frac{x_{i+1}-x}{h_i} \right)$$

$$+ \left[y(x_{i+1}) - \frac{h_i^2 M_{i+1}}{6} \right]\left(\frac{x-x_i}{h_i} \right)$$

$$x_i \le x \le x_{i+1}, \quad i = 0(1)N-1 \qquad (5.5)$$

where $M_i = S''(x_i)$, $i=0(1)N$

Now, consider the following difference scheme for computing the approximate solution $y_1, y_2, \ldots, y_{N-1}$ at the nodal points $x_1, x_2, \ldots, x_{N-1}$.

$$- \varepsilon\, a_{0i} y_{i-1} - \varepsilon\, a_{1i} y_i - \varepsilon\, y_{i+1}$$

$$+ \varepsilon\, h_i^2 \left[B_{0i} y''_{i-\lambda} + B_{1i} y''_i + B_{2i} y''_{i+\lambda} \right] = 0 \qquad (5.6)$$

$$i = 1(1)N-1$$

with

$$y_0 = y(a) = \alpha, \quad y_N = y(b) = \beta . \qquad (5.7)$$

where $\frac{1}{2} < \lambda \le 1$,

The coefficient a's and B's are determined by using Taylor's series expansion and equating the coefficients of $h_i^r y^{(r)}(x_i)$, $r = 0(1)4$ to zero.

Thus, we have the following set of five equations:

$$- a_{0i} + a_{1i} - 1 = 0 \tag{5.8}$$

$$- \frac{a_{0i}}{\sigma_i} - 1 = 0 \tag{5.9}$$

$$- \frac{1}{2\sigma_i^2} a_{0i} - \frac{1}{2} + (B_{0i} + B_{1i} + B_{2i}) = 0 \tag{5.10}$$

$$\frac{1}{6\sigma_i^3} a_{0i} - \frac{1}{6} - \frac{\lambda}{\sigma} B_{0i} + \lambda B_{2i} = 0 \tag{5.11}$$

$$- \frac{1}{24\sigma_i^2} a_{0i} - \frac{1}{24} + \frac{\lambda^2}{2\sigma_i^2} B_{0i} + \frac{\lambda^2}{2} B_{2i} = 0 \tag{5.12}$$

where $\sigma_i = h_i/h_{1-1}$.

From (5.8)-(5.12), we have

$$a_{0i} = \sigma_i \tag{5.13}$$

$$a_{1i} = 1 + \sigma_i \tag{5.14}$$

$$B_{0i} = \frac{\sigma_i(\sigma_i - 1)(1-2\lambda) + 1}{12 \lambda^2 \sigma_i} \tag{5.15}$$

$$B_{1i} = \frac{(\sigma_i^3 + 1)(2\lambda-1) + 2\lambda(3\lambda-1)\sigma_i(\sigma_i+1) + (2\lambda-1)}{12 \lambda^2 \sigma_i^2} \tag{5.16}$$

and

$$B_{2i} = \frac{\sigma_i^2 + (2\lambda-1)(\sigma_i - 1)}{12\ \lambda^2\ \sigma_i^2} \tag{5.17}$$

For $\varepsilon\ y_i''$ and $\varepsilon\ y_{i\pm\lambda}''$, we take the following approximations

$$\varepsilon\ y_i'' = p_i y_i + q_i \tag{5.18}$$

and $\varepsilon\ y_{i\pm\lambda}'' = p_{i\pm\lambda}\bar{S}(x_{i\pm\lambda}) + q_{i\pm\lambda}$ (5.19)

where $p_{i\pm\lambda} = p(x_{i\pm\lambda})$, $p_i = p(x_i)$, $q_{i\pm\lambda} = q(x_{i\pm\lambda})$, $q_i = q(x_i)$,

and $\bar{S}(x_{i\pm\lambda})$ are obtained by putting $x = x_i+\lambda h_i$ and $x = x_i-\lambda h_{i-1}$

in $\bar{S}(x)$ given by

$$\bar{S}(x) = \frac{(x_{i+1}-x)^3}{6h_i}\bar{M}_i + \frac{(x-x_i)^3}{6h_i}\bar{M}_{i+1} + \left[y_i - \frac{h_i^2\bar{M}_i}{6}\right]\left(\frac{x_{i+1}-x}{h_i}\right)$$

$$+ \left[y_{i+1} - \frac{h_i^2\bar{M}_{i+1}}{6}\right]\left(\frac{x-x_i}{h_i}\right)$$

$$x_i \le x \le x_{i+1},\ i = 0(1)N-1 \tag{5.20}$$

where

$$\varepsilon\ \bar{M}_j = p(x_j)y_j + q(x_j) \qquad j = i, i+1 \tag{5.21}$$

Putting the values of $\varepsilon\ y_i''$ and $\varepsilon\ y_{i\pm\lambda}''$ in (5.6) along with values of a's and B's, we get the following tridiagonal system which gives the approximate solution $y_1, y_2, \ldots, y_{N-1}$ at the nodal points $x_1, x_2, \ldots x_{N-1}$

$$\left[-\varepsilon\sigma_i + h_i^2 B_{0i} p_{i-\lambda}\left(\lambda + p_{i-1} a_1(\lambda,h_{i-1})\right)\right] y_{i-1}$$

$$+ \left[\varepsilon(1+\sigma_i) + h_i^2 p_i B_{1i} + h_i^2(1-\lambda)\left(B_{0i} p_{i-\lambda} + B_{2i} p_{i+\lambda}\right)\right.$$

$$\left. + h_i^2 p_i\left(B_{0i} p_{i-\lambda} a_2(\lambda,h_{i-1}) + B_{2i} p_{i+\lambda} a_2(\lambda,h_i)\right)\right] y_i$$

$$+ \left[-\varepsilon + h_i^2 B_{2i} p_{i+\lambda}\left(\lambda + p_{i+1} a_1(\lambda,h_i)\right)\right] y_{i+1}$$

$$= -h_i^2 \left\{a_1(\lambda,h_{i-1}) B_{0i} p_{i-\lambda} q_{i-1} + a_1(\lambda,h_i) B_{2i} p_{i+\lambda} q_{i+1}\right.$$

$$+ a_2(\lambda,h_{i-1}) B_{0i} p_{i-\lambda} q_i + a_2(\lambda,h_i) B_{2i} p_{i+\lambda} q_i$$

$$\left. + B_{0i} q_{i-\lambda} + B_{2i} q_{i+\lambda}\right\} - h_i^2 B_{1i} q_i$$

$$i = 1(1)N-1 \qquad (5.22)$$

$$y_0 = y(a) = \alpha, \quad y_N = y(b) = \beta, \qquad (5.23)$$

where $a_1(\lambda,h) = \frac{h^2\lambda(\lambda^2-1)}{6\varepsilon}$, $a_2(\lambda,h) = \frac{h^2(1-\lambda)}{6\varepsilon}[(1-\lambda)^2-1]$.

Remark 5.1

For $\varepsilon = 1$ (regular problem), $\sigma_i \equiv 1$ $\forall$ i (uniform mesh) and $\lambda=1$, we have from (5.22)

$$\left(-1 + \frac{h^2}{12} p_{i-1}\right) y_{i-1} + \left(2 + \frac{10h^2}{12} p_i\right) y_i + \left(-1 + \frac{h^2}{12} p_{i+1}\right) y_{i+1}$$

$$= -\frac{h^2}{12}\left(q_{i-1} + 10q_i + q_{i+1}\right) \qquad i = 1(1)N-1 \qquad (5.24)$$

$y_0 = \alpha$, $y_N = \beta$

which is a well known fourth order Numerov's method for the regular problem

$$y'' = p(x)y + q(x) \tag{5.25}$$

5.3 SEMI-LINEAR PROBLEMS :

For semi-linear problems of the form (5.1)-(5.2), we use the quasi-linearization technique. Starting from the initial guess $u_0 = u_0(x)$, we obtain the sequence of linear boundary value problems

$$\varepsilon u''_{n+1} = f_y(x,u_n)\, u_{n+1} + f(x,u_n) - f_y(x,u_n)u_n \tag{5.26}$$

$$u_{n+1}(a) = \alpha, \quad u_{n+1}(b) = \beta \tag{5.27}$$

It is known that (Doolan et.al. 1980), if $\dfrac{\partial^2 f(x,y)}{\partial y^2}$ is bounded for all $x \in [a,b]$ and all real y, then there exists a constant C, independent of n and ε, s.t.

$$\left| u_{n+1}(x) - u(x) \right| \le C \left| u_n(x) - u(x) \right|^2 \tag{5.28}$$

where $u(x)$ is the solution of (5.1)-(5.2).

We note that each linearized eq. (5.26) can be written in the form (5.3) by putting

$$p(x) = f_y(x,u_n) \quad \text{and} \quad q(x) = f(x,u_n) - f_y(x,u_n)u_n \tag{5.29}$$

In order to exclude turning points in (5.26)-(5.27), we assume that, for each $n \ge 0$, a constant K_n exists such that

$$f_y(x,u_n) \ge K_n > 0, \quad x \in [a,b] \tag{5.30}$$

5.4 MESH SELECTION PROCEDURE :

For simplicity we take $\sigma_i = \sigma$ ($\forall$ i).

Since there are boundary layers at both ends, we first consider first half of the interval [a, (a + b)/2] with $\sigma > 1$, Given the values of N (an even positive number) and σ, we can choose h_0 from (4.26) of previous chapter as

$$h_0 = \frac{(b-a)(\sigma-1)}{2(\sigma^{N/2}-1)} \qquad \text{for } \sigma > 1 \tag{5.31}$$

and subsequent h_i's can be obtained as $h_i = \sigma h_{i-1}$, i=1,N/2-1

Then we take its mirror image in other half of the interval i.e.,[(a + b)/2, b]. This ensures more mesh points in the boundary layer regions at both ends.

5.5 ERROR ANALYSIS :

In matrix-vector form, the tridiagonal system (5.22) can be written as

$$MY = D \tag{5.32}$$

where $M = (m_{ij})$, $1 \le i,j \le N-1$ is a tridiagonal matrix with

$$m_{i,i+1} = -\varepsilon + h_i^2 B_{2i} p_{i+\lambda} \left(\lambda + p_{i+1} a_1(\lambda,h_1)\right) \qquad i = 1, N-2$$

$$m_{i,i} = \varepsilon(1+\sigma_i) + h_i^2 p_i B_{1i} + h_i^2(1-\lambda)\left(B_{0i} p_{i-\lambda} + B_{2i} p_{i+\lambda}\right)$$

$$+ h_i^2 p_1\left[B_{0i} p_{i-\lambda} a_2(\lambda,h_{i-1}) + B_{2i} p_{i+\lambda} a_2(\lambda,h_i)\right] \qquad i = 1, N-1$$

$$m_{i,i-1} = -\varepsilon\,\sigma_i + h_i^2\, B_{0i}\, p_{i-\lambda}\,[\lambda + p_{i-1}\, a_1(\lambda,\, h_{i-1})]$$

$$i = 2,\ N-2$$

and $D = (d_i)$, $1 \le i \le N-1$ is a column vector with

$$d_i = -h_i^2[a_1(\lambda,h_{i-1})\, B_{0i}\, p_{i-\lambda}\, q_{i-1} + a_1(\lambda,h_i)\, B_{2i}\, p_{i+\lambda}\, q_{i+1}$$

$$+ a_2(\lambda,h_{i-1})\, B_{0i}\, p_{i-\lambda}\, q_{i-1} + a_2(\lambda,h_i)\, B_{2i}\, p_{i+\lambda}\, \lambda\, q_i$$

$$+ B_{0i}\, q_{i-\lambda} + B_{2i}\, q_{i+\lambda} + B_1 q_i]$$

and $Y = (y_1, y_2, \ldots, y_{N-1})^t$.

Also, we have

$$MY_A - T(h) = D \tag{5.33}$$

where $Y_A = (y(x_1), y(x_2), \ldots, y(x_{N-1}))^t$ denotes the actual solution and

$T(h) = \left(T(h_1), \ldots, T(h_{N-1})\right)^t$ is the local truncation error with

$T(h_i)$ given by

$$T(h_i) = \frac{\varepsilon}{360\sigma_i^5}\left[(5\lambda-3)(\sigma_i^5-\sigma_i) + 5\lambda(2\lambda-1)(\sigma_i^4-\sigma_i^2)\right]h_i^5 y^{(v)}(x_i) + 0(h_i^6) \tag{5.34}$$

From (5.32) and (5.33), we have

$$M(Y_A - Y) = T(h) \tag{5.35}$$

Thus, the error equation is

$$ME = T(h) \tag{5.36}$$

where $E = Y_A - Y$.

From (5.15), it can be seen that B_{0i}'s are positive for $\frac{1}{2} < \lambda \le 1$, if σ_i satisfies

$$\sigma_i(\sigma_i - 1)(1-2\lambda) + 1 > 0 \tag{5.37}$$

which implies,

$$(2\lambda-1)\, \sigma_i^2 - (2\lambda-1)\, \sigma_i - 1 < 0 \tag{5.38}$$

which is equivalent to

$$\left[\sigma_i - \left(\frac{1}{2} + \frac{1}{2}\sqrt{\frac{3+2\lambda}{2\lambda-1}}\right)\right]\left[\sigma_i - \left(\frac{1}{2} - \frac{1}{2}\sqrt{\frac{3+2\lambda}{2\lambda-1}}\right)\right] < 0 \tag{5.39}$$

Therefore, either

$$\sigma_i < \left(\frac{1}{2} + \frac{1}{2}\sqrt{\frac{3+2\lambda}{2\lambda-1}}\right) \quad \text{and} \quad \sigma_i > \left(\frac{1}{2} - \frac{1}{2}\sqrt{\frac{3+2\lambda}{2\lambda-1}}\right) \tag{5.40}$$

or

$$\sigma_i > \left(\frac{1}{2} + \frac{1}{2}\sqrt{\frac{3+2\lambda}{2\lambda-1}}\right) \quad \text{and} \quad \sigma_i < \left(\frac{1}{2} - \frac{1}{2}\sqrt{\frac{3+2\lambda}{2\lambda-1}}\right) \tag{5.41}$$

Since $\sigma_i > 0$, second inequality in condition (5.40) is always satisfied and second inequality in condition (5.41) does not hold always, we must have

$$\sigma_i < \left(\frac{1}{2} + \frac{1}{2}\sqrt{\frac{3+2\lambda}{2\lambda-1}}\right) \tag{5.42}$$

Similarly, from (5.17) it can be seen that B_{2i}'s are positive for $\frac{1}{2} < \lambda \leq 1$, if σ_i satisfies

$$\sigma_i^2 + (2\lambda-1)(\sigma_i - 1) > 0 \tag{5.43}$$

which is equivalent to

$$\left[\sigma_i - (2\lambda-1)\left(-\frac{1}{2} + \frac{1}{2}\sqrt{\frac{3+2\lambda}{2\lambda-1}}\right)\right]\left[\sigma_i - (2\lambda-1)\left(-\frac{1}{2} - \frac{1}{2}\sqrt{\frac{3+2\lambda}{2\lambda-1}}\right)\right] > 0 \tag{5.44}$$

Therefore, either

$$\sigma_i > (2\lambda-1)\left(-\frac{1}{2} + \frac{1}{2}\sqrt{\frac{3+2\lambda}{2\lambda-1}}\right) \quad \text{and} \quad \sigma_i > (2\lambda-1)\left(-\frac{1}{2} - \frac{1}{2}\sqrt{\frac{3+2\lambda}{2\lambda-1}}\right) \tag{5.46}$$

or

$$\sigma_i < (2\lambda-1)\left(-\frac{1}{2} + \frac{1}{2}\sqrt{\frac{3+2\lambda}{2\lambda-1}}\right) \quad \text{and} \quad \sigma_i < (2\lambda-1)\left(-\frac{1}{2} - \frac{1}{2}\sqrt{\frac{3+2\lambda}{2\lambda-1}}\right) \tag{5.47}$$

Since $\sigma_i > 0$, second inequality in condition (5.46) is always satisfied and second inequality in condition (5.47) does not hold always, we must have

$$\sigma_i > (2\lambda-1)\left(-\frac{1}{2} + \frac{1}{2}\sqrt{\frac{3+2\lambda}{2\lambda-1}}\right) \tag{5.48}$$

Combining conditions (5.43) and (5.48), we have

$$(2\lambda-1)\left(-\frac{1}{2} + \frac{1}{2}\sqrt{\frac{3+2\lambda}{2\lambda-1}}\right) < \sigma_i < \left(\frac{1}{2} + \frac{1}{2}\sqrt{\frac{3+2\lambda}{2\lambda-1}}\right) \tag{5.49}$$

For the tridigonal matrix M to be irreducibly diagonally dominant, we do the follwing if $m_{i,i-1}$ and $m_{i,i+1}$ are non-positve.

for $m_{i,i-1} < 0$, we must have,

$$- \varepsilon \sigma_i + h_i^2 B_{0i} p_{i-\lambda} [\lambda + p_{i-1} a_1(\lambda, h_{i-1})] < 0 \tag{5.50}$$

Now, if

$$h_i^2 B_{0i} p_{i-\lambda} [\lambda + p_{i-1} a_1(\lambda, h_{i-1})] < 0 \tag{5.51}$$

then (5.50) is satisfied

Substituting the value of $a_1(\lambda,h_{i-1})$ we have

$$h_i^2 B_{0i} p_{i-\lambda} \left[\lambda + p_{i-1} \frac{h_{i-1}^2}{6\varepsilon} \lambda(\lambda^2-1)\right] < 0 \tag{5.52}$$

which is equivalent to

$$h_i^2 B_{0i} p_{1-\lambda} \lambda + \frac{h_i^4}{6\varepsilon\sigma_i^2} \lambda(\lambda^2-1) B_{0i} p_{i-\lambda} p_{i-1} < 0 \tag{5.53}$$

which implies

$$6\varepsilon\sigma_i^2 < h_i^2 p_{i-1}(1 - \lambda^2) \tag{5.54}$$

Similarly, for $m_{i,i+1} < 0$, we must have,

$$- \varepsilon + h_1^2 B_{2i} p_{i+\lambda} \left(\lambda + p_{i+1} a_1 (\lambda,h_i)\right) < 0 \tag{5.55}$$

Now, if

$$h_1^2 B_{2i} p_{i+\lambda} \left(\lambda + p_{i+1} a_1 (\lambda,h_i)\right) < 0 \tag{5.56}$$

then condition (5.55) is satisfied

Substituting the value of $a_1(\lambda,h_i)$, we have,

$$h_i^2\ B_{2i}\ p_{i+\lambda}\ \left[\lambda + p_{i+1}\frac{h_i^2}{6\varepsilon}\ \lambda(\lambda^2-1)\right] < 0 \qquad (5.57)$$

which is equivalent to,

$$h_i^2\ B_{2i}\ p_{i+\lambda}\ \lambda + \frac{h_i^4}{6\varepsilon} - \lambda(\lambda^2-1)\ B_{2i}\ p_{i+\lambda}\ p_{i+1} < 0 \qquad (5.58)$$

which implies,

$$6\varepsilon < h_i^2\ p_{i+1}(1 - \lambda^2) \qquad (5.59)$$

Let $\underline{\mathcal{P}} = \min\limits_{a\le x\le b} p(x)$, then condition (5.54) and condition (5.59) is satisfied if

$$6\varepsilon\sigma_i^2 < h_i^2\ \underline{\mathcal{P}}\ (1-\lambda^2) \qquad (5.60)$$

For $m_{i,i} > 0$, we must have

$$\varepsilon(1+\ \sigma_1) + h_i^2\ p_1\ B_{1i} + h_i^2(1-\lambda)\left(B_{0i}\ p_{i-\lambda} + B_{2i}\ p_{i+\lambda}\right)$$
$$+ h_i^2\ p_1\left[B_{0i}\ p_{i-\lambda}\ a_2(\lambda,h_{i-1}) + B_{2i}\ p_{i+\lambda}\ a_2(\lambda,h_i)\right] > 0 \qquad (5.61)$$

It is enough to have,

$$h_i^2\ p_i\ B_{1i} + h_i^2(1-\lambda)\left(B_{0i}\ p_{i-\lambda} + B_{2i}\ p_{i+\lambda}\right)$$
$$+ h_i^2\ p_i\left[B_{0i}\ p_{i-\lambda}\ a_2(\lambda,h_{i-1}) + B_{2i}\ p_{i+\lambda}\ a_2(\lambda,h_i)\right] > 0 \qquad (5.62)$$

which is equivalent to,

$$p_i\ B_{1i} + (1-\lambda)\left(B_{0i}\ p_{i-\lambda} + B_{2i}\ p_{i+\lambda}\right)$$

$$+ p_i\left[B_{0i}\, p_{i-\lambda}\, a_2(\lambda,h_{i-1}) + B_{2i}\, p_{i+\lambda}\, a_2(\lambda,h_i)\right] > 0 \tag{5.63}$$

Substituting the value of $a_2(\lambda,h_{i-1})$ and $a_2(\lambda,h_i)$, we have

$$p_i\, B_{1i} + (1-\lambda)\left(B_{0i}\, p_{i-\lambda} + B_{2i}\, p_{i+\lambda}\right) >$$

$$p_iB_{0i}\, p_{i-\lambda}\, \frac{h_{i-1}^2}{6\varepsilon}\, (1-\lambda)[1-(1-\lambda^2)] + p_iB_{2i}\, p_{i+\lambda}\, \frac{h_i^2}{6\varepsilon}\, (1-\lambda)[1-(1-\lambda^2)] \tag{5.64}$$

Let $\overline{\mathcal{P}} = \max\limits_{a\le x\le b} p(x)$, then it is equivalent to

$$\underline{\mathcal{P}}\left(B_{1i} + (1-\lambda)(B_{0i} + B_{2i})\right) >$$

$$\overline{\mathcal{P}}^2\left(B_{0i}\, \frac{h_i^2}{6\varepsilon\sigma_i^2}(1-\lambda)[1-(1-\lambda^2)] + B_{2i}\, \frac{h_i^2}{6\varepsilon}\, (1-\lambda)[1-(1-\lambda^2)]\right) \tag{5.65}$$

which is equivalent to

$$\frac{6\varepsilon\sigma_i^2\underline{\mathcal{P}}\left(B_{1i} + (1-\lambda)(B_{0i} + B_{2i})\right)}{\overline{\mathcal{P}}^2\lambda(1-\lambda)(2-\lambda)\left(B_{0i} + \sigma_i^2B_{2i}\right)} > h_i^2 \tag{5.66}$$

For diagonal dominance, we must have

$$|\, m_{i,i}\,| - |\, m_{i,i-1}\,| - |\, m_{i,i+1}\,| > 0 \tag{5.67}$$

Substituting these values, we must have

$$\varepsilon(1+\sigma_i) + h_i^2\, p_i\, B_{1i} + h_i^2(1-\lambda)\left(B_{0i}\, p_{i-\lambda} + B_{2i}\, p_{i+\lambda}\right)$$

$$+ h_i^2 p_i\left[B_{0i} p_{i-\lambda} a_2(\lambda,h_{i-1}) + B_{2i} p_{i+\lambda} a_2(\lambda,h_1)\right]$$

$$- \varepsilon \sigma_i + h_i^2 B_{0i} p_{i-\lambda} [\lambda + p_{i-1} a_1(\lambda, h_{i-1})]$$

$$- \varepsilon + h_i^2 B_{2i} p_{i+\lambda} \left(\lambda + p_{i+1} a_1 (\lambda,h_i)\right) > 0 \tag{5.68}$$

Which is equivalent to

$$h_i^2 \left(p_{i-\lambda} B_{0i} + p_i B_{1i} + p_{i+\lambda} B_{2i} \right) >$$

$$\frac{h_i^4}{6\varepsilon\sigma_i^2} \left[\lambda(1-\lambda^2)\left(B_{0i} p_i p_{i-\lambda} +\sigma_i^2 B_{2i} p_{i+1}p_{i+\lambda}\right)\right.$$

$$\left. + \lambda(1-\lambda)(2-\lambda)\left(B_{0i} p_i p_{i-\lambda} + \sigma_i^2 B_{2i} p_{i+1}p_{i+\lambda}\right) \right] \tag{5.69}$$

Therefore, it is enough to have

$$\underline{\mathcal{P}} \left(B_{0i} + B_{1i} + B_{2i} \right) > \frac{h_i^2}{2\varepsilon\sigma_i^2} \bar{\mathcal{P}}^2 \lambda(1-\lambda^2)\left(B_{0i} + \sigma_i^2 B_{2i} \right) \tag{5.70}$$

i.e.,

$$2\varepsilon\sigma_i^2 \underline{\mathcal{P}} \left(B_{0i} + B_{1i} + B_{2i} \right) > h_i^2 \bar{\mathcal{P}}^2 \lambda(1-\lambda^2)\left(B_{0i} + \sigma_i^2 B_{2i} \right) \tag{5.71}$$

which implies

$$\frac{2\varepsilon\sigma_i^2 \underline{\mathcal{P}} \left(B_{0i} + B_{1i} + B_{2i} \right)}{\bar{\mathcal{P}}^2 \lambda(1-\lambda^2)\left(B_{0i} + \sigma_i^2 B_{2i}\right)} > h_i^2 \tag{5.72}$$

Combining conditions (5.60),(5.66), and (5.72), we have the following condition on h_i for M to be irreducilbly diagonal dominant and hence monotone,

$$H_1 < h_i^2 < H_2 H_3 \tag{5.73}$$

where

$$H_1 = \sigma_i^2 \frac{6\varepsilon}{\underline{\mathcal{P}}(1-\lambda^2)}$$

$$H_2 = \frac{\varepsilon\sigma_i^2 \underline{\mathcal{P}}}{\bar{\mathcal{P}}^2 \lambda(1-\lambda)\left(B_{0i} + \sigma_i^2 B_{2i}\right)}$$

and

$$H_3 = \min\left[\frac{6\left(B_1 + (1-\lambda)(B_{0i} + B_{2i})\right)}{(2-\lambda)}, \frac{2\left(B_{1i} + B_{0i} + B_{2i}\right)}{(1+\lambda)} \right]$$

from eq. (5.36)

$$E = M^{-1} T(h) \tag{5.74}$$

Let $S_i = \sum_{j=1}^{N-1} m_{ij}$, where m_{ij} is the $(i,j)^{th}$ element of the matrix M.

Then

$$S_i = h_i^2 \Big[B_{0i} p_{i-\lambda} \Big(\lambda + p_{i-1}\, a_1(\lambda,\, h_{i-1})\Big)$$

$$+ p_i\, B_{1i} + (1-\lambda)(B_{0i}\, p_{i-\lambda} + B_2\, p_{i+\lambda})$$

$$+ p_i\Big(B_{0i}\, p_{i-\lambda}\, a_2(\lambda,h_{i-1}) + B_{2i}\, p_{i+\lambda}\, a_2(\lambda,h_i)\Big)$$

$$+ B_{2i}\, p_{i+\lambda} \Big(\lambda + p_{i+1}\, a_1(\lambda,h_i)\Big)\Big] \qquad (5.75)$$

For h_i satisfying (5.73), we have $S_i > 0$, which implies

$$S_i \geq h_i^2\, A_i > 0 \qquad (5.76)$$

where $A_i = B_{0i}\, p_{i-\lambda} + p_i\, B_{1i} + B_{2i}\, p_{i+\lambda} > 0$

Also, $$\sum_{i=1}^{N-1} m_{k,i}^{-1}\, S_i = 1 \qquad k = 1(1)N-1 \qquad (5.77)$$

where $m_{k,i}^{-1}$ is the $(k,i)^{th}$ element of the matrix M^{-1}.

Therefore

$$\sum_{i=1}^{N-1} m_{k,i}^{-1} \leq \frac{1}{\min\limits_{1 \leq i \leq m} S_i} = \frac{1}{h_{i_0}^2\, A_{i_0}} \qquad (5.78)$$

for some i_0 between 1 and N-1

From (5.74) we have

$$e_j = \sum_{i=1}^{N-1} m_{j,i}^{-1}\, T(h_i) \qquad (5.79)$$

which implies

$$|e_j| \leq \sum_{i=1}^{N-1} m_{j,i}^{-1}\, |T(h_i)| \qquad j = 1(1)N-1 \qquad (5.80)$$

with the help of (5.34), (5.78) and (5.80), it follows that

$$|e_i| = O(h_i^3) \tag{5.81}$$

i.e.

$$\| E \| = O(h^3) \tag{5.82}$$

where $h = \max_i h_i$.

In addition, if $\sigma_i \equiv 1$ (uniform mesh), (5.34) gives

$$T(h_i) = O(h_i^6) \tag{5.83}$$

Therefore, from (5.80), we have

$$\| E \| = O(h^4) \tag{5.84}$$

where $h = \max_i h_i$, i.e. our methods reduces to a family of fourth order methods for uniform mesh.

5.6 NUMERICAL EXAMPLES :

Example 5.1 (Doolan et. al. 1980)

$$\varepsilon y'' = y + \cos^2 \pi x + 2\pi^2 \varepsilon \cos 2\pi x$$

$$y(0) = y(1) = 0$$

The exact solution is given by

$$y(x) = \frac{\left(e^{-(1+x)/\sqrt{\varepsilon}} + e^{-x/\sqrt{\varepsilon}}\right)}{\left(1 + e^{-1/\sqrt{\varepsilon}}\right)} - \cos^2 \pi x \, .$$

Example 5.2 (Carrier 1970)

$$\varepsilon y'' = (2 - x)^2 y - 1$$

$$y(-1) = y(1) = 0$$

We have taken Carrier's approximate solution for comparison purpose which is given by

with the help of (5.34), (5.78) and (5.80), it follows that

$$|e_i| = O(h_i^3) \tag{5.81}$$

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5.6 NUMERICAL EXAMPLES :

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Example 5.2 (Carrier 1970)

$$\varepsilon y'' = (2 - x)^2 y - 1$$

$$y(-1) = y(1) = 0$$

We have taken Carrier's approximate solution for comparison purpose which is given by

$$y(x) = (2 - x^2)^{-1} - \exp[-(x+1)/\sqrt{\varepsilon}] - \exp[-(1-x)/\sqrt{\varepsilon}]$$

<u>Example 5.3</u> (Carrier 1970)

$$\varepsilon y'' = 1 - 2b(1-x^2)y - y^2$$
$$y(-1) = y(1) = 0$$

This example is solved for b = 0.0 and 1.0 by Ascher and Wiess (1984), for b = 0.5 by Roberts (1986). We have tabulated results for b = 0.0, 0.5 and 1.0 and taken Carrier's approximate solution for comparison purpose, which is given by

$$y(x) = y_R(x) + 12\exp(p_1)/[1 + \exp(p1)]^2$$
$$+ 12\exp(p_2)/[1 + \exp(p2)]^2$$

where $y_R(x)$ is the solution of the reduced problem given by

$$y_R(x) = -b(1-x^2) - [b^2(1-x^2)^2 + 1]^{1/2}$$

and

$$p_{1,2} = (2/\varepsilon)^{1/2}(1 \pm x) + 2\log(\sqrt{2} + \sqrt{3})$$

Using the quasi-linearization method, we have

$$\varepsilon u''_{n+1} = [-2b(1-x^2) - 2u_n)u_{n+1} + u_n^2 + 1$$
$$u_{n+1}(0) = u_{n+1}(1) = 0$$

The initial guess $u_0(x)$ is taken as

$u_0(0) = u_0(1) = 0$ and $u_0(x) = y_R(x)$ elsewhere and stopping criteria is

$$\max | u_{n+1}(i) - u_n(i) | \le \xi \quad \forall i$$

where ξ is a given constant.

5.7 DISCUSSION:

We have implemented our method on three examples. Maximum errors at the nodal points, i.e. $\max_i | y_i - y(x_i)|$ are tabulated in Tables 5.1-5.3 for different values of ε, N, λ and σ.

The second order method developed for general linear singularly perturbed boundary value problems using continuity condition can be applied here, but it will only give second order accuracy. Comparison between the results of previous chapter and this chapter for example 1 shows that error is reduced because of high order approximations.

Table 5.1

Absolute maximum error in example 5.1

	N	σ	λ = 0.85	λ = 0.90	λ = 1.00
$\varepsilon=10^{-5}$	60	1.00	8.44233 E -2	2.62515 E -2	5.75677 E -2
		1.05	2.27848 E -3	2.53310 E -3	1.08140 E -2
		1.15	9.50132 E -5	9.98950 E -5	1.14810 E -4
	120	1.00	2.26680 E -3	4.90570 E -3	1.74408 E -2
		1.05	3.75959 E -5	1.69450 E -5	1.10074 E -5
		1.15	2.99077 E -5	8.12125 E -5	2.15746 E -5
$\varepsilon=10^{-8}$	100	1.00	1.34668 E +1	3.52120	1.00872
		1.05	7.76747 E -1	6.63801 E -1	9.08492 E -2
		1.15	3.40082 E -4	3.31122 E -4	3.35928 E -4
	200	1.00	1.09771	1.29771	1.00427 E -1
		1.05	8.35841 E -4	1.46160 E -3	5.47756 E -3
		1.15	2.47516 E -5	1.24349 E -5	1.51054 E -5
$\varepsilon=10^{-10}$	150	1.00	3.43029 E +1	1.34162	1.01017 E -1
		1.05	1.12917 E +1	8.63424 E -1	9.30022 E -2
		1.15	4.20789 E -5	5.57506 E -5	5.46106 E -5
	300	1.00	1.00075	3.87275	1.01007 E -1
		1.05	4.85113 E -4	8.80132 E -4	3.28161 E -3
		1.15	2.20034 E -5	5.55807 E -5	1.49807 E -5
$\varepsilon=10^{-12}$	200	1.00	2.23670	2.15699	1.01020 E -1
		1.05	6.87588	1.21240	9.32885 E -2
		1.15	3.03721 E -5	1.99388 E -5	2.27321 E -5
	400	1.00	1.67117	1.74504	1.01020 E -1
		1.05	3.21882 E -4	4.89111 E -4	1.92046 E -3
		1.15	3.02942 E -5	1.98478 E -5	1.50451 E -5

Table 5.2

Absolute maximum error in example 5.2

	N	σ	λ = 0.85	0.90	1.00
$\varepsilon=10^{-5}$	120	1.00	8.27090 E -2	2.53335 E -2	5.61221 E -2
		1.05	1.28270 E -3	1.38051 E -3	1.55133 E -3
		1.15	1.37861 E -3	1.37754 E -3	1.37500 E -3
	240	1.00	1.16652 E -3	5.90189 E -3	1.82053 E -2
		1.05	1.38983 E -3	1.38973 E -3	1.38970 E -3
		1.15	1.38116 E -3	1.38026 E -3	1.37815 E -3
$\varepsilon=10^{-8}$	200	1.00	1.02338	2.16152	9.89045 E -2
		1.05	2.19239 E -2	8.88709 E -4	3.26500 E -2
		1.15	3.48469 E -5	3.39509 E -5	3.20879 E -5
	400	1.00	1.09756	9.29963 E -1	9.94366 E -2
		1.05	4.18056 E -5	4.17778 E -5	4.17120 E -5
		1.15	3.49235 E -5	3.40341 E -5	3.22187 E -5
$\varepsilon=10^{-10}$	300	1.00	8.68514 E -1	9.70422	9.96923 E -2
		1.05	1.14491 E -2	2.22162 E -3	2.47163 E -2
		1.15	8.87946 E -6	9.72226 E -6	1.22260 E -5
	600	1.00	9.19354 E -1	9.85432 E -1	1.00339 E -1
		1.05	4.08246 E -6	4.06773 E -6	4.03352 E -6
		1.15	8.87869 E -6	9.72143 E -6	1.22250 E -5
$\varepsilon=10^{-12}$	400	1.00	1.63941	1.62333	1.00022 E -1
		1.05	5.94386 E -3	3.02194 E -3	1.81159 E -2
		1.15	1.10274 E -5	1.20722 E -5	1.47466 E -5
	800	1.00	3.46654	2.76668	1.00518 E -1
		1.05	1.10274 E -5	1.20721 E -5	1.47466 E -5
		1.15	3.23433 E -7	6.41523 E -6	2.85756 E -7

Table 5.3(a)

Absolute maximum error in example 5.3($b = 0$)

	N	σ	λ = 0.85	0.90	1.00
$\varepsilon=10^{-5}$	120	1.05	1.61162 E -3	1.96165 E -3	2.74121 E -3
		1.10	7.18562 E -6	7.49952 E -6	8.57211 E -6
		1.15	1.22547 E -5	1.30979 E -5	1.54331 E -5
	240	1.05	4.12840 E -7	4.32614 E -7	4.98784 E -7
		1.10	2.43147 E -6	2.60394 E -6	3.10189 E -6
		1.15	1.12607 E -5	1.20405 E -5	1.43034 E -5
$\varepsilon=10^{-8}$	200	1.05	5.63527 E -3	7.80762 E -3	1.12728 E -2
		1.10	5.29125 E -6	5.57571 E -6	6.45412 E -6
		1.15	1.13972 E -5	1.22301 E -5	1.44895 E -5
	400	1.05	2.94392 E -7	3.11404 E -7	3.64386 E -7
		1.10	2.41536 E -6	2.59594 E -6	3.09548 E -6
		1.15	1.12856 E -5	1.21109 E -5	1.43491 E -5
$\varepsilon=10^{-10}$	300	1.05	6.88183 E -4	5.76028 E -3	4.78037 E -3
		1.10	2.59192 E -6	2.78538 E -6	3.31420 E -6
		1.15	1.12046 E -5	1.20324 E -5	1.43744 E -5
	600	1.05	1.61166 E -7	1.73394 E -7	2.07105 E -7
		1.10	2.41532 E -6	2.60003 E -6	3.09477 E -6
		1.15	1.12035 E -5	1.20314 E -5	1.43732 E -5
$\varepsilon=10^{-12}$	400	1.05	6.97031 E -4	2.18612 E -3	8.98200 E -4
		1.10	1.15156 E -5	2.35158 E -5	3.10965 E -6
		1.15	1.12824 E -5	3.06506 E -5	1.43658 E -5
	800	1.05	5.49768 E -6	5.33567 E -6	1.97697 E -7
		1.10	1.10224 E -5	2.95868 E -5	3.09112 E -6
		1.15	3.26594 E -5	3.35720 E -5	1.43658 E -5

Table 5.3(b)

Absolute maximum error in example 5.3(b = 0.5)

	N	σ	λ = 0.85	0.90	1.00
$\varepsilon=10^{-5}$	120	1.05	1.61062 E -3	1.96065 E -3	2.74020 E -3
		1.10	6.50927 E -6	6.58689 E -6	7.52364 E -6
		1.15	1.14087 E -5	1.22518 E -5	1.45869 E -5
	240	1.05	6.63025 E -6	9.32859 E -6	6.50359 E -6
		1.10	6.50710 E -6	6.57344 E -6	6.52460 E -6
		1.15	1.04508 E -5	1.12305 E -5	1.33829 E -5
$\varepsilon=10^{-8}$	200	1.05	5.63527 E -3	7.80762 E -3	1.12728 E -2
		1.10	5.29012 E -6	5.57458 E -6	6.45299 E -6
		1.15	1.13962 E -5	1.22291 E -5	1.44886 E -5
	400	1.05	2.93323 E -7	3.10335 E -7	3.63255 E -7
		1.10	2.41442 E -6	2.59489 E -6	3.09443 E -6
		1.15	1.12846 E -5	1.21099 E -5	1.43481 E -5
$\varepsilon=10^{-10}$	300	1.05	6.88182 E -4	5.76028 E -3	4.78037 E -3
		1.10	2.59191 E -6	1.02274 E -5	3.31419 E -6
		1.15	1.12046 E -5	1.20324 E -5	1.43744 E -5
	600	1.05	1.61652 E -7	1.73384 E -7	2.07095 E -7
		1.10	2.41530 E -6	9.29281 E -6	3.09476 E -6
		1.15	1.12035 E -5	1.20314 E -5	1.43731 E -5
$\varepsilon=10^{-12}$	400	1.05	6.97031 E -4	2.18612 E -3	8.98200 E -4
		1.10	5.37324 E -6	4.06362 E -5	3.10966 E -6
		1.15	1.13545 E -5	1.21148 E -5	1.43658 E -5
	800	1.05	9.87236 E -7	2.75375 E -6	1.97699 E -7
		1.10	2.42010 E -6	2.16882 E -5	3.09112 E -6
		1.15	1.12824 E -5	3.89226 E -5	1.43658 E -5

Table 5.3(c)

Absolute maximum error in example 5.3(b = 1.0)

	N	σ	λ = 0.85	λ = 0.90	λ = 1.00
$\varepsilon=10^{-5}$	120	1.05	1.59898 E -3	1.94900 E -3	2.72854 E -3
		1.10	3.37277 E -4	1.21724 E -5	1.22124 E -5
		1.15	1.21174 E -5	1.21216 E -5	1.22741 E -5
	240	1.05	1.98066 E -5	2.95402 E -5	1.45904 E -5
		1.10	5.71400 E -5	1.33572 E -5	1.30403 E -5
		1.15	1.21174 E -5	1.21218 E -5	1.22741 E -5
$\varepsilon=10^{-8}$	200	1.05	5.63526 E -3	7.80763 E -3	1.12728 E -2
		1.10	5.27643 E -6	5.56089 E -6	6.43930 E -6
		1.15	1.13830 E -5	1.22158 E -5	1.44754 E -5
	400	1.05	2.79844 E -7	2.96857 E -7	3.49581 E -7
		1.10	2.40139 E -6	2.58148 E -6	3.08101 E -6
		1.15	1.12714 E -5	1.20967 E -5	1.43349 E -5
$\varepsilon=10^{-10}$	300	1.05	6.88183 E -4	5.76028 E -3	4.78037 E -3
		1.10	2.59197 E -6	2.80660 E -6	3.31405 E -6
		1.15	1.12044 E -5	1.20323 E -5	1.43742 E -5
	600	1.05	1.61522 E -7	6.20482 E -7	2.06962 E -7
		1.10	2.41517 E -6	3.01709 E -6	3.09462 E -6
		1.15	1.12034 E -5	1.20313 E -5	1.43730 E -5
$\varepsilon=10^{-12}$	400	1.05	6.97031 E -4	2.18612 E -3	8.98200 E -4
		1.10	1.15156 E -5	2.35158 E -5	3.10965 E -6
		1.15	1.12824 E -5	3.06506 E -5	1.43658 E -5
	800	1.05	5.49768 E -6	5.33567 E -6	1.97697 E -7
		1.10	1.10224 E -5	2.95868 E -5	3.09112 E -6
		1.15	3.26594 E -5	3.35720 E -5	1.43658 E -5

CHAPTER VI

Third Order Variable Mesh Methods For Non-linear Singularly Perturbed Boundary Value Problems

6.1 INTRODUCTION :

Consider the following class of general non-linear singularly perturbed boundary value problems

$$\varepsilon y'' = f(x,y,y') \qquad a < x < b,\ 0 < \varepsilon << 1 \tag{6.1}$$

$$y(a) = A,\ y(b) = B \tag{6.2}$$

where we assume that f is a smooth function satisfying:

(i) $\frac{\partial}{\partial z} f(x,y,z) \leq 0$

(ii) $\frac{\partial}{\partial y} f(x,y,z) > 0$

(iii) $(\frac{\partial}{\partial y} - \frac{\partial}{\partial z}) f(x,y,z) \geq \alpha > 0$

and (iv) the growth condition

$$f(x,y,z) = O(|z|^2) \quad \text{as } z \longrightarrow \infty$$

$$\text{for all } (x,y,z) \in [a,b] \times \mathbb{R}^2$$

Under these conditions, the problem (6.1)-(6.2) has a unique solution, (Doolan et. al 1980)

These problems occur in many areas of engineering and applied mathematics, notably fluid mechanics. Many practical areas where these types of problems occur and the difficulties associated in solving them have been discussed in previous chapters.

In this chapter, the methods developed in the previous chapter for semi-linear problems are generalised for the non-linear problems of the above form. A two parameter family of third order variable mesh methods using cubic spline approximations is obtained, which gives more accurate third order approximations for first derivative terms also. These third order variable mesh methods reduce to a family of fourth order methods when the mesh ratio σ_i is taken to be unity. Also when both σ_i and λ are taken to be unity, a well known Numerov method is obtained as a particular case for the given problem with $\varepsilon = 1$ and f independent of y'.

6.2 DERIVATION OF THIRD ORDER METHODS :

Let $x_0 = a$, $x_i = a + \sum_{k=0}^{i-1} h_k$, $i=1(1)N-1$ $h_k = x_{k+1} - x_k$, $x_N = b$.

Given the values of $y(x_0), y(x_1), \ldots, y(x_N)$ of a function $y(x)$ at the nodal points $x_0, x_1, \ldots, x_N$, the interpolating cubic spline can be given in the following form (Ahlberg et.al. 1967)

$$S(x) = \frac{(x_{i+1} - x)^3}{6h_i} M_i + \frac{(x - x_i)^3}{6h_i} M_{i+1} + \left[y(x_i) - \frac{h_i^2 M_i}{6} \right]\left(\frac{x_{i+1} - x}{h_i}\right)$$

$$+ \left[y(x_{i+1}) - \frac{h_i^2 M_{i+1}}{6} \right]\left(\frac{x - x_i}{h_i}\right)$$

$$x_i \le x \le x_{i+1} \;, \; i = 0(1)N-1 \tag{6.3}$$

where $M_i = S''(x_i)$

and its derivative is given by

$$S'(x) = \left[\frac{(x - x_i)^2}{2h_i} - \frac{h_i}{6}\right]M_{i+1} + \left[\frac{-(x_{i+1} - x)^2}{2h_i} + \frac{h_i}{6}\right]M_i$$

$$+ \frac{y(x_{i+1}) - y(x_i)}{h_i} \qquad x_i \le x \le x_{i+1}\ ,\ i = 0(1)N-1 \qquad (6.4)$$

Now consider the following difference scheme for computing the approximate solution $y_1, y_2, \ldots, y_{N-1}$ at the nodal points $x_1, x_2, \ldots, x_{N-1}$.

$$-\varepsilon\, a_{0i} y_{i-1} + \varepsilon\, a_{1i} y_i - \varepsilon\, y_{i+1} + h_i^2 (B_{0i}\, f_{i-\lambda} + B_{1i}\, f_i + B_{2i}\, f_{i+\lambda}) = 0$$

$$i = 1(1)N-1 \qquad 1/\sqrt{3} < \lambda \le 1 \qquad (6.5)$$

where f_i and $f_{i\pm\lambda}$ are approximations of $f(x_i, y(x_i), y'(x_i))$ and $f(x_{i\pm\lambda}, y(x_{i\pm\lambda}), y'(x_{i\pm\lambda}))$ respectively. The coefficient a's and B's are determined by replacing the approximate values by exact values and equating the coefficients of $h_i^r y^{(r)}(x_i)$, $r = 0(1)4$ to zero after taking Taylor's series expansion.

Thus, as in previous chapter, we have

$$a_{0i} = \sigma_i \qquad (6.6)$$

$$a_{1i} = 1 + \sigma_i \qquad (6.7)$$

$$B_{0i} = \frac{\sigma_i(\sigma_i - 1)(1-2\lambda) + 1}{12\lambda^2 \sigma_i} \qquad (6.8)$$

$$B_{1i} = \frac{(\sigma_i^3 + 1)(2\lambda - 1) + 2\sigma_i(\sigma_i + 1)\lambda(3\lambda - 1)}{12\lambda^2 \sigma_i^2} \qquad (6.9)$$

$$\text{and} \qquad B_{2i} = \frac{\sigma_i^2 + (\sigma_i - 1)(2\lambda - 1)}{12\lambda^2 \sigma_i^2} \tag{6.10}$$

where $\sigma_i = h_i/h_{i-1}$, $\qquad i=1(1)N-1$

Consider the following approximations of the first derivatives at the nodal points $x_1, x_2, \ldots, x_{N-1}$ in terms of $y_1, y_2, \ldots, y_{N-1}$.

$$y'_{i-1} = \frac{1}{h_i(1+\sigma_i)} \left[-y_{i+1} + (\sigma_i + 1)^2 y_i - \sigma_i(\sigma_i + 2) y_{i-1} \right] \qquad i=1(1)N-1 \tag{6.11}$$

$$y'_{i+1} = \frac{1}{h_i(1+\sigma_i)} \left[(2\sigma_i + 1) y_{i+1} - (\sigma_i + 1)^2 y_i + \sigma_i^2 y_{i-1} \right] \qquad i=1(1)N-1 \tag{6.12}$$

$$\text{and } y'_i = \frac{1}{h_i(1+\sigma_i)} \left[y_{i+1} + (\sigma_i^2 - 1) y_i - \sigma_i^2 y_{i-1} \right] - \frac{h_i}{6(\sigma_i+1)\varepsilon} \left[f_{i+1} - f_{i-1} \right] \qquad i=1(1)N-1 \tag{6.13}$$

Now we define f_i and $f_{i\pm\lambda}$ as

$$f_i = f(x_i, y_i, y'_i) \tag{6.14}$$

$$f_{i\pm\lambda} = f(x_{i\pm\lambda}, \bar{S}(x_{i\pm\lambda}), \bar{S}'(x_{i\pm\lambda})) \qquad i=1(1)N-1 \tag{6.15}$$

where $\bar{S}(x_{i\pm\lambda})$ and $\bar{S}'(x_{i\pm\lambda})$ are obtained by putting $x = x_i + \lambda h_i$ and $x_i - \lambda h_{i-1}$ in $\bar{S}(x)$ and $\bar{S}'(x)$, that are given by

$$\bar{S}(x) = \frac{(x_{i+1} - x)^3}{6h_i} \bar{M}_i + \frac{(x - x_i)^3}{6h_i} \bar{M}_{i+1} + \left[y_i - \frac{h_i^2 \bar{M}_i}{6} \right] \left(\frac{x_{i+1} - x}{h_i} \right)$$

$$+ \left[y_{i+1} - \frac{h_i^2 \bar{M}_{i+1}}{6} \right] \left(\frac{x - x_i}{h_i} \right)$$

$$x_i \le x \le x_{i+1}, \quad i = 0(1)N-1 \tag{6.16}$$

$$\bar{S}'(x) = \left[\frac{(x - x_i)^2}{2h_i} - \frac{h_i}{6}\right]\bar{M}_{i+1} + \left[\frac{-(x_{i+1} - x)^2}{2h_i} + \frac{h_i}{6}\right]\bar{M}_i$$

$$+ \frac{y_{i+1} - y_i}{h_i} \qquad x_i \le x \le x_{i+1}, \quad i = 0(1)N-1 \qquad (6.17)$$

where $\varepsilon \bar{M}_j = f(x_j, y_j, y'_j)$, $\qquad j = i, i+1$

After obtaining the values of $f_i, f_{i\pm\lambda}$ from Eqs.(6.14)-(6.15) and putting the values of a's from Eqs.(6.6)-(6.7) in Eq(6.5), we can write the finite difference scheme in the following form

$$-\varepsilon\, \sigma_i y_{i-1} + \varepsilon(1+\sigma_i) y_i - \varepsilon\, y_{i+1} + h_i^2 (B_{0i}\, f_{i-\lambda} + B_{1i}\, f_i + B_{2i}\, f_{i+\lambda}) = 0, \qquad (6.18)$$

with

$$y_0 = y(a) = A\,, \quad y_N = y(b) = B$$

where B's are given by Eqs.(6.8)-(6.10).

If the differential equation is linear, then the resulting tridiagonal system can be solved by Thomas algorithm, in the non-linear case, the system can be solved by quasi-linearization technique (Doolan et al.,1980) described in section 6.3.

Remark 6.1 For $\varepsilon = \lambda = \sigma_i = 1 \;\; \forall\, i$, our finite difference scheme (6.18) reduces to well known Numerov's method for the problem $y'' = f(x,y)$, which is given as

$$-y_{i-1} + 2y_i - y_{i+1} + \frac{h^2}{12}\left[f(x_{i-1}, y_{i-1}) + 10\, f(x_i, y_i) + f(x_{i+1}, y_{i+1})\right] = 0 \qquad (6.19)$$

6.3 NON-LINEAR PROBLEMS :

For non-linear problems of the form (6.1)-(6.2), we use the quasi-linearization technique. Starting from the initial guess $u_0 = u_0(x)$, we obtain the sequence of linear boundary value problems

$$\varepsilon u''_{n+1} = f_{y'}(x,u_n,u'_n)\, u'_{n+1} + f_y(x,u_n,u'_n)u_{n+1} + f(x,u_n,u'_n)$$
$$- f_{y'}(x,u_n,u'_n)u'_n - f_y(x,u_n,u'_n)u_n \tag{6.20}$$

$$u_{n+1}(a) = \alpha, \quad u_{n+1}(b) = \beta \tag{6.21}$$

It is known that (Doolan et.al. 1980), if $\frac{\partial^2 f(x,y,z)}{\partial y^2}$ and $\frac{\partial^2 f(x,y,z)}{\partial y^2}$ are bounded for all $x \in [a,b]$ and all real y and z.Then if $\frac{\partial^2 f}{\partial y'^2} \equiv 0$, there exists a constant C, independent of n and ε, s.t.

$$\left|u_{n+1}(x) - u(x)\right| \le C \left|u_n(x) - u(x)\right| \tag{6.22}$$

where $u(x)$ is the solution of (6.1)-(6.2).

6.4 ERROR ANALYSIS :

Substituting for y_i, the actual solution $y(x_i)$ in Eq (6.18), we have

$$-\varepsilon\, \sigma_i y(x_{i-1}) + \varepsilon(1+\sigma_i)y(x_i) - \varepsilon y(x_i)$$
$$+ h_i^2 \Big[B_{0i} f(x_{i-\lambda},y(x_{i-\lambda}),y'(x_{i-\lambda})) + B_{1i}\; f(x_i,y(x_i,y(x_i),y'(x_i))$$
$$+ B_{2i}\; f(x_{i+\lambda},y(x_{i+\lambda}),y'(x_{i+\lambda}))\Big] = T(h_i) \qquad i=1(1)N-1 \tag{6.23}$$

where $T(h_i)$ is the local truncation error and defined by

$$T(h_i) = \frac{\varepsilon}{360\sigma_i^5}\left[(5\lambda - 3)(\sigma_i^5 - \sigma_i) + 5\lambda(2\lambda - 1)(\sigma_i^4 - \sigma_i^2)\right]h_i^5 y^{(V)}(x_i)$$

$$+ O(h_i^6) \qquad (6.24)$$

Subtracting Eq(6.18) from Eq(6.23) and denoting $e_i = y(x_i) - y_i$, we have

$$-\varepsilon\, \sigma_i e_{i-1} + \varepsilon\, (1+\sigma_i)\, e_i - \varepsilon\, e_{i+1}$$

$$+ h_i^2 \Big\{B_{0i}\left[f(x_{i-\lambda}, y(x_{i-\lambda}), y'(x_{i-\lambda})) - f(x_{i-\lambda}, \bar{S}(x_{i-\lambda}), \bar{S}'(x_{i-\lambda}))\right]$$

$$+ B_{1i}\left[f(x_i, y(x_i), y'(x_i)) - f(x_i, y_i, y_i')\right]$$

$$+ B_{2i}\left[f(x_{i+\lambda}, y(x_{i+\lambda}), y'(x_{i+\lambda})) - f(x_{i+\lambda}, \bar{S}(x_{i+\lambda}), \bar{S}'(x_{i+\lambda}))\right]\Big\}$$

$$= T(h_i) \qquad (6.25)$$

Now let

$$y(x_{i\pm\lambda}) - \bar{S}(x_{i\pm\lambda}) = (y(x_{i\pm\lambda}) - S(x_{i\pm\lambda})) + (S(x_{i\pm\lambda}) - \bar{S}(x_{i\pm\lambda}))$$

$$= e_{I\pm} + e_{D\pm} \quad \text{(say)} \qquad (6.26)$$

$$y'(x_{i\pm\lambda}) - \bar{S}'(x_{i\pm\lambda}) = (y'(x_{i\pm\lambda}) - S'(x_{i\pm\lambda})) + (S'(x_{i\pm\lambda}) - \bar{S}'(x_{i\pm\lambda}))$$

$$= e'_{I\pm} + e'_{D\pm} \qquad (6.27)$$

and $$y'(x_j) - y_j' = e_j' \qquad j = i, i\pm 1 \qquad (6.28)$$

From the theory of splines, it is known that(Ahlberg et. al. 1967)

$$e_{I\pm} = C_\pm h_i^4 + O(h_i^5) \qquad (6.29)$$

and $$e'_{I\pm} = C'_\pm h_i^3 + O(h_i^4) \qquad (6.30)$$

where $C_{\pm}$ are constants independent of h_i and

$C'_{\pm}$ are constants independent of h_i and has $\sigma_i - 1$ as a factor i.e.,

$C'_{\pm} = D'_{\pm}(\sigma_i - 1)$ where $D'_{\pm}$ are constants independent of h_i.

Since, the approximations $y'_{i\pm1}$ of $y'(x_{i+1})$ are taken to be same as in chapter 4,(see eq. 4.32 & 4.34) we know that these are second order approximations, therefore we take

$$e'_{i-1} = C_1 h_i^2 + O(h_i^3) \tag{6.31}$$

$$e'_{i+1} = C_2 h_i^2 + O(h_i^3) \tag{6.32}$$

where

$$C_1 = \frac{h_i^2}{2} y'''(\xi_3^{(i)}) + \frac{h_i^2}{6}\left[\left(\frac{2+\sigma_i}{1+\sigma_i}\right)y'''(\xi_1^{(i)}) + \left(\frac{\sigma_i - 1}{\sigma_i^2(\sigma_i+1)}\right)y'''(\xi_2^{(i)})\right]$$

and $$C_2 = \frac{h_i^2}{2\sigma_i^2} y'''(\xi_4^{(i)}) + \frac{h_i^2}{6}\left[\left(-\frac{1}{\sigma_i(1+\sigma_i)}\right)y'''(\xi_1^{(i)}) + \left(\frac{2\sigma_i+1}{\sigma_i^2(\sigma_i+1)}\right)y'''(\xi_2^{(i)})\right]$$

For finding $e'_i = y'_i - y'(x_i)$, we do the following.

Taking the usual Taylor series expansion for y around x_i, we get the following expressions for y_{i+1} and y_{i-1}:

$$y_{i+1} = y_i + h_i y'_i + \frac{h_i^2}{2} y''_i + \frac{h_i^3}{6} y'''_i + \frac{h_i^4}{24} y^{iv}(\xi_5^{(i)}) \tag{6.33}$$

$$y_{i-1} = y_i - h_{i-1}y_i' + \frac{h_{i-1}^2}{2} y_i'' - \frac{h_{i-1}^3}{6} y_i''' + \frac{h_{i-1}^4}{24} y^{iv}(\xi_6^{(i)})$$

$$x_{i-1} < \xi_5^{(i)},\ \xi_6^{(i)} < x_{i+1}$$

(6.34)

Multiplying (6.34) by σ_i^2, and subtracting it from (6.33), we get the following expression

$$y_{i+1} + (\sigma_i^2-1)y_i - \sigma_i^2 y_{i-1} = h_i(1+\sigma_i)y_i' + \frac{h_i^3}{6\sigma_i}(1+\sigma_i)y_i'''$$

$$+ \frac{h_i^4}{24}\left(y^{iv}(\xi_5^{(i)}) - \frac{1}{\sigma_i^2} y^{iv}(\xi_6^{(i)}) \right)$$

(6.35)

which implies

$$\frac{y_{i+1} + (\sigma_i^2-1)y_i - \sigma_i^2 y_{i-1}}{h_i(1+\sigma_i)} = y_i' + \frac{h_i^2}{6\sigma_i} y_i'''$$

$$+ \frac{h_i^3}{24(1+\sigma_i)}\left(y^{iv}(\xi_5^{(i)}) - \frac{1}{\sigma_i^2} y^{iv}(\xi_6^{(i)})\right)$$

(6.36)

Also, we have

$$y_{i+1}'' = y_i'' + h_i y_i''' + \frac{h_i^2}{2} y^{iv}(\xi_5^{(i)}) \qquad (6.37)$$

and $$y_{i-1}'' = y_i'' - h_{i-1}y_i''' + \frac{h_{i-1}^2}{2} y^{iv}(\xi_6^{(i)}) \qquad (6.38)$$

which implies,

$$y_{i+1}'' - y_{i-1}'' = h_i(1 + \frac{1}{\sigma_i})\ y_i''' + \frac{h_i^2}{2}\left(y^{iv}(\xi_5^{(i)}) - \frac{1}{\sigma_i^2} y^{iv}(\xi_6^{(i)})\right)$$

(6.39)

Therefore,

$$\frac{h_i}{6(\sigma_i+1)\varepsilon}\left[f_{i+1}-f_{i-1}\right] = \frac{h_i}{6(\sigma_i+1)}\left[y''_{i+1}-y''_{i-1}\right]$$

$$= \frac{h_i^2}{6\sigma_i}y'''_i + \frac{h_i^3}{12(1+\sigma_i)}\left(y^{iv}(\xi_5^{(i)}) - \frac{1}{\sigma_i^2}y^{iv}(\xi_6^{(i)})\right) \tag{6.40}$$

Now subtracting this equation from (6.36), we have

$$\frac{y_{i+1} + (\sigma_i^2-1)y_i - \sigma_i^2 y_{i-1}}{h_i(1+\sigma_i)} - \frac{h_i}{6(\sigma_i+1)\varepsilon}\left[f_{i+1}-f_{i-1}\right]$$

$$= y'_i - \frac{h_i^3}{24(1+\sigma_i)}\left(y^{iv}(\xi_5^{(i)}) - \frac{1}{\sigma_i^2}y^{iv}(\xi_6^{(i)})\right) \tag{6.41}$$

which imlpies,

$$y'_i = y'(x_i) - \frac{h_i^3}{24(1+\sigma_i)}\left(y^{iv}(\xi_5^{(i)}) - \frac{1}{\sigma_i^2}y^{iv}(\xi_6^{(i)})\right) \tag{6.42}$$

which shows that y'_i is third order approximation to $y'(x_i)$, therefore we can write

$$e'_i = C_3 h_i^3 + O(h_i^4) \tag{6.43}$$

where

$$C_3 = -\frac{h_i^3}{24(1+\sigma_i)}\left(y^{iv}(\xi_5^{(i)}) - \frac{1}{\sigma_i^2}y^{iv}(\xi_6^{(i)})\right)$$

From mean value theorem, we have

$$f(x_j, y(x_j), y'(x_j)) - f(x_j, y_j, y'_j) = \left.\frac{\partial f}{\partial y}\right|_{x=x_j} (y(x_j) - y_j) + \left.\frac{\partial f}{\partial y'}\right|_{x=x_j} (y'(x_j) - y'_j)$$

$$= u_j e_j + v_j e'_j, \quad j = i, i \pm 1 \tag{6.44}$$

where $\left.\frac{\partial f}{\partial y}\right|_{x=x_j}$ and $\left.\frac{\partial f}{\partial y'}\right|_{x=x_j}$ are denoted by u_j and v_j respectively.

Subtracting (6.16) from (6.3), we have

$$S(x) - \overline{S}(x)$$

$$= \left[\frac{(x_{i+1} - x)^3}{6h_i} - \frac{h_i (x_{i+1} - x)}{6} \right] (M_i - \overline{M}_i) + \frac{x_{i+1} - x}{h_i} \Big(y(x_i) - y_i \Big)$$

$$+ \left[\frac{(x - x_i)^3}{6h_i} - \frac{h_i (x - x_i)}{6} \right] (M_{i+1} - \overline{M}_{i+1}) + \frac{x - x_i}{h_i} \Big(y(x_{i+1}) - y_{i+1} \Big) \tag{6.45}$$

Substituting $\varepsilon M_j = f(x_j, y(x_j), y'(x_j))$

and $\varepsilon \overline{M}_j = f(x_j, y_j, y'_j), \quad j = i, i+1$

and applying the mean value theorem, we have

$$S(x) - \overline{S}(x) = \left[\frac{(x_{i+1} - x)^3}{6h_i \varepsilon} - \frac{h_i (x_{i+1} - x)}{6\varepsilon} \right] \Big(u_i e_i + v_i e'_i \Big) + \frac{x_{i+1} - x}{h_i} e_i$$

$$+ \left[\frac{(x - x_i)^3}{6h_i \varepsilon} - \frac{h_i (x - x_i)}{6\varepsilon} \right] \Big(u_{i+1} e_{i+1} + v_{i+1} e'_{i+1} \Big) + \frac{x - x_i}{h_i} e_{i+1} \tag{6.46}$$

Putting $x = x_i + \lambda h_i$, we get

$$e_{D+} = S(x_i+\lambda h_i) - \bar{S}(x_i+\lambda h_i)$$

$$= \frac{h_i^2}{6\varepsilon} \lambda(\lambda^2-1)\left[u_{i+1}e_{i+1} + v_{i+1}e'_{i+1} \right]$$

$$+ \frac{h_i^2}{6\varepsilon} (1-\lambda)((1-\lambda)^2-1)\left[u_i e_i + v_i e'_i \right] + \lambda e_{i+1} + (1-\lambda)e_i \quad (6.47)$$

Similarly putting $x = x_i - \lambda h_{i-1}$, we get

$$e_{D-} = S(x_i-\lambda h_{i-1}) - \bar{S}(x_i-\lambda h_{i-1})$$

$$= \frac{h_i^2}{6\varepsilon\sigma_i^2} \lambda(\lambda^2-1) \left[u_{i-1}e_{i-1} + v_{i-1}e'_{i-1} \right]$$

$$+ \frac{h_i^2}{6\varepsilon\sigma_i^2}(1-\lambda)((1-\lambda)^2-1)\left[u_i e_i + v_i e'_i\right] + \lambda e_{i-1} + (1-\lambda)e_i$$

$$(6.48)$$

Subtracting (6.17) from (6.4), we have

$$S'(x) - \bar{S}'(x)$$

$$= \left[\frac{-(x_{i+1}- x)^2}{2h_i} + \frac{h_i}{6} \right](M_i - \bar{M}_i) + \frac{y(x_{i+1})- y_{i+1}}{h_i}$$

$$+ \left[\frac{(x - x_i)^2}{2h_i} - \frac{h_i}{6} \right](M_{i+1} - \bar{M}_{i+1}) - \frac{y(x_i) - y_i}{h_i}$$

$$(6.49)$$

Again applying mean value theorem and substituting as above, we have

$S'(x) - \bar{S}'(x)$

$$= \left[\frac{-(x_{i+1} - x)^2}{2h_i\varepsilon} + \frac{h_i}{6\varepsilon} \right] \left(u_i e_i + v_i e_i' \right)$$

$$+ \left[\frac{(x - x_i)^2}{2h_i\varepsilon} - \frac{h_i}{6\varepsilon} \right] \left(u_{i+1} e_{i+1} + v_{i+1} e_{i+1}' \right) + \frac{e_{i+1} - e_i}{h_i} \tag{6.50}$$

Putting $x = x_i + \lambda h_i$, we get

$$e_{D+}' = \frac{e_{i+1} - e_i}{h_i} + \frac{h_i}{2\varepsilon}(\lambda^2 - 1/3) \left[u_{i+1} e_{i+1} + v_{i+1} e_{i+1}' \right]$$

$$+ \frac{h_i}{2\varepsilon}(1/3 - (1-\lambda)^2) \left[u_i e_i + v_i e_i' \right] \tag{6.51}$$

Similarly putting $x = x_i - \lambda h_{i-1}$, we get

$$e_{D-}' = S'(x_i - \lambda h_{i-1}) - \bar{S}'(x_i - \lambda h_{i-1})$$

$$= \frac{\sigma_i(e_i - e_{i-1})}{h_i} - \frac{h_i}{2\varepsilon\sigma_i}(\lambda^2 - 1/3) \left[u_{i-1} e_{i-1} + v_{i-1} e_{i-1}' \right]$$

$$- \frac{h_i}{2\varepsilon\sigma_i}(1/3 - (1-\lambda)^2) \left[u_i e_i + v_i e_i' \right] \tag{6.52}$$

Applying mean value theorem for the expressions in Eq(6.25), we have

$$-\varepsilon\ \sigma_i e_{i-1} + \varepsilon(1+\sigma_i)e_i - \varepsilon\ e_{i+1}$$

$$+ h_i^2 \left\{ B_{0i}\left[\left.\frac{\partial f}{\partial y}\right|_{x=x_i-\lambda h_{i-1}} \left(y(x_{i-\lambda}) - \bar{s}(x_{i-\lambda})\right) \right.\right.$$

$$\left. + \left.\frac{\partial f}{\partial y'}\right|_{x=x_i-\lambda h_{i-1}} \left(y'(x_{i-\lambda}) - \bar{s}'(x_{i-\lambda})\right) \right]$$

$$+ B_{1i}\left[\left.\frac{\partial f}{\partial y}\right|_{x=x_i} \left(y(x_i) - y_i\right) + \left.\frac{\partial f}{\partial y'}\right|_{x=x_i} \left(y'(x_i) - y_i'\right) \right]$$

$$+ B_{2i}\left[\left.\frac{\partial f}{\partial y}\right|_{x=x_i+\lambda h_i} \left(y(x_{i+\lambda}) - \bar{s}(x_{i+\lambda})\right) \right.$$

$$\left.\left. + \left.\frac{\partial f}{\partial y'}\right|_{x=x_i+\lambda h_i} \left(y'(x_{i+\lambda}) - \bar{s}'(x_{i+\lambda})\right) \right] \right\} = T(h_i) \tag{6.53}$$

Therefore, we have

$$-\varepsilon\ \sigma_i e_{i-1} + \varepsilon(1+\sigma_i)e_i - \varepsilon\ e_{i+1}$$

$$+ h_i^2 \left\{ B_{0i}\left[u_{i-\lambda}\ (e_{I-} + e_{D-}) + v_{i-\lambda}\ (e'_{I-} + e'_{D-}) \right] \right.$$

$$+ B_{1i}\left[u_i\ e_i + v_i\ e_i' \right]$$

$$\left. + B_{2i}\left[u_{i+\lambda}\ (e_{I+} + e_{D+}) + v_{i+\lambda}\ (e'_{I+} + e'_{D+}) \right] \right\} = T(h_i) \tag{6.54}$$

where

$$u_{i-\lambda} = \left.\frac{\partial f}{\partial y}\right|_{x=x_i-\lambda h_{i-1}},\quad u_{i+\lambda} = \left.\frac{\partial f}{\partial y}\right|_{x=x_i+\lambda h_i}$$

and $$v_{i-\lambda} = \left.\frac{\partial f}{\partial y'}\right|_{x=x_i-\lambda h_{i-1}},\quad v_{i+\lambda} = \left.\frac{\partial f}{\partial y'}\right|_{x=x_i+\lambda h_i}$$

Putting the values of $e_{I\pm}, e'_{I\pm}, e_{D\pm}$ and $e'_{D\pm}$ and e'_j $(j=i,i\pm1)$ from (6.29)-(6.32), (6.43), (6.47), (6.48), (6.51), (6.52), we get

$$-\varepsilon\sigma_i\ e_{i-1} + \varepsilon(1+\sigma_i)\ e_i - \varepsilon\ e_{i+1}$$

$$+ h_i^2 \Bigg\{ B_{0i} \Bigg[u_{1-\lambda} \Bigg(C_- h_i^4 + \frac{h_i^2}{6\varepsilon\sigma_i^2} \lambda(\lambda^2-1) \Big[u_{i-1}e_{i-1} + v_{i-1}\ C_1\ h_i^2 \Big]$$

$$+ \frac{h_i^2}{6\varepsilon\sigma_i^2} (1-\lambda)((1-\lambda)^2-1)\Big[u_i e_i + v_i C_3 h_i^3 \Big] + \lambda\ e_{i-1} + (1-\lambda)e_i \Bigg)$$

$$+ v_{i-\lambda}\Bigg(C'_- h_1^3 + \frac{\sigma_i(e_i-e_{i-1})}{h_i} - \frac{h_i}{2\varepsilon\sigma_i}(\lambda^2 - 1/3)\Big[u_{i-1}e_{i-1} + v_{i-1}C_1 h_i^2\Big]$$

$$- \frac{h_i}{2\varepsilon\sigma_i}(1/3 - (1-\lambda)^2)\Big[u_i e_i + v_i C_3 h_i^3 \Big] \Bigg) \Bigg]$$

$$+ B_{1i} \Big[u_i e_i + v_i C_3 h_i^3 \Big]$$

$$+ B_{2i} \Bigg[u_{i+\lambda} \Bigg(C_+ h_i^4 + \frac{h_i^2}{6\varepsilon} \lambda(\lambda^2 -1) \Big[u_{i+1}\ e_{i+1} + v_{i+1}\ C_2\ h_i^3 \Big]$$

$$+ \frac{h_1^2}{6\varepsilon} (1-\lambda)((1-\lambda)^2-1)\Big[u_1 e_i + v_i C_i h_i^3 \Big] + \lambda e_{i+1}\ (1-\lambda)e_i \Bigg)$$

$$+ v_{i+\lambda} \Bigg(C'_+ h_1^3 + \frac{e_{1+1} - e_i}{h_1} + \frac{h_i}{2\varepsilon}(\lambda^2-1/3)\Big[u_{i+1}e_{i+1} + v_{i+1}C_2 h_i^2 \Big]$$

$$+ \frac{h_i}{2\varepsilon}(1/3 - (1-\lambda)^2)\Big[u_i e_i + v_i C_3 h_1^3 \Big] \Bigg) \Bigg] \Bigg\} + O(h_1^6)$$

$$= T(h_i) \tag{6.55}$$

Clubbing the coefficients of e_{i-1}, e_i and e_{i+1} and shifting all otther terms to right hand side, we get

$$\Big[-\varepsilon\,\sigma_i + \frac{h_i^4}{6\varepsilon\sigma_i^2} B_{0i}\lambda(\lambda^2-1)u_{i-\lambda}u_{i-1} + h_i^2 B_{0i}u_{i-\lambda}\,\lambda$$

$$- h_i\sigma_i B_{0i}v_{i-\lambda} - \frac{h_i^3}{2\varepsilon\sigma_i} B_{0i}v_{i-\lambda}u_{i-1}(\lambda^2 - 1/3)\Big]e_{i-1}$$

$$+ \Big[\,\varepsilon(1+\sigma_i) + \frac{h_i^4}{6\varepsilon\sigma_i^2} B_{0i}(1-\lambda)[(1-\lambda^2)-1]u_i u_{i-\lambda} + h_i^2(1-\lambda)B_{0i}u_{i-\lambda}$$

$$+ h_i B_{0i}\sigma_i v_{i-\lambda} - \frac{h_i^3}{2\varepsilon\sigma_i} B_{0i}v_{i-\lambda}(1/3 - (1-\lambda)^2)u_i + h_i^2 B_{1i}u_i$$

$$+ \frac{h_i^4}{6\varepsilon} B_{2i}(1-\lambda)[(1-\lambda)^2 1]u_i u_{i+\lambda} + h_i^2(1-\lambda)B_{2i}u_{i+\lambda} - h_i B_{2i}v_{i+\lambda}$$

$$+\frac{h_i^3}{2\varepsilon\sigma_i} B_{2i}v_{i+\lambda}(1/3 - (1-\lambda)^2)u_i\,\Big]\,e_i$$

$$+ \Big[-\varepsilon + \frac{h_i^4}{6\varepsilon} B_{2i}\lambda(\lambda^2-1)u_{i+\lambda}u_{i+1} + h_i^2 B_{2i}u_{i+\lambda}\,\lambda + h_i B_{2i}v_{i+\lambda}$$

$$+ \frac{h_i^3}{2\varepsilon} B_{2i}v_{i+\lambda}u_{i+1}(\lambda^2 - 1/3)\Big]e_{i+1}$$

$$= T(h_i) - T_0(h_i)$$

$$= \bar{T}(h_i) \quad \text{(say)} \qquad i=1(1)N-1 \qquad (6.56)$$

where $T_0(h_1) =$

$$h_i^2 \left\{ B_{0i} \left[u_{i-\lambda} \left(C_- h_i^4 + \frac{h_i^4}{6\varepsilon\sigma_i^2} \lambda(\lambda^2-1)\ v_{i-1}\ C_1 \right.\right.\right.$$

$$+ \frac{h_i^5}{6\varepsilon\sigma_i^2} (1-\lambda)((1-\lambda)^2-1)\ v_i C_3 h_i^3$$

$$+ v_{i-\lambda}\Bigg(C'_- h_1^3 - \frac{h_i^3}{2\varepsilon\sigma_1} (\lambda^2 - 1/3)\ v_{i-1}C_1$$

$$\left.\left. - \frac{h_i^4}{2\varepsilon\sigma_i} (1/3 - (1-\lambda)^2\ v_i C_3 \right) \right]$$

$$+ B_{1i}\ v_i C_3 h_i^3$$

$$+ B_{2i} \left[u_{i+\lambda} \left(C_+ h_i^4 + \frac{h_i^4}{6\varepsilon} \lambda(\lambda^2 -1)\ v_{i+1}\ C_2 \right.\right.$$

$$\left. + \frac{h_i^5}{6\varepsilon} (1-\lambda)((1-\lambda)^2-1)\ v_i C_3 \right)$$

$$+ v_{i+\lambda} \left(C'_+ h_1^3 + \frac{h_i^3}{2\varepsilon}(\lambda^2-1/3) + v_{i+1}C_2 \right.$$

$$\left.\left.\left. + \frac{h_1^3}{2\varepsilon} (1/3 - (1-\lambda)^2)\ v_i C_3 \right) \right] \right\} + O(h_i^6)$$

Clearly,

$$T_0(h_i) = O(h_i^5) \tag{6.57}$$

Also, from (6.21)

$$T(h_i) = O(h_i^5) \tag{6.58}$$

Therefore,

$$\overline{T}(h_i) = O(h_i^5) \tag{6.59}$$

Putting Eq(6.56) in the matrix-vector form,we have

$$ME = \bar{T} \tag{6.60}$$

where M is an (N-1)x(N-1) tridiagonal matrix whose elements are given by

$$m_{i,i+1} = \text{coefficient of } e_{i+1} \text{ in } (6.56) \qquad i=1(1)N-2$$
$$m_{i,i} = \text{coefficient of } e_{i} \text{ in } (6.56) \qquad i=1(1)N-1$$
$$m_{i,i-1} = \text{coefficient of } e_{i-1} \text{ in } (6.56) \qquad i=2(1)N-1$$

$$E = (e_1,e_2,\ldots,e_{N-1})^{tr} \quad ,\bar{T} = (\bar{T}(h_1),\bar{T}(h_2),\ldots,\bar{T}(h_{N-1}))^{tr}$$

As in previous chapter, it can be seen that B_{0i}, B_{1i} and B_{2i} are positive for $1/2 < \lambda \le 1$ if σ_i satisfies

$$(2\lambda-1)\left(-\frac{1}{2}+\frac{1}{2}\sqrt{\frac{3+2\lambda}{2\lambda-1}}\right) < \sigma_i < \left(\frac{1}{2}+\frac{1}{2}\sqrt{\frac{3+2\lambda}{2\lambda-1}}\right) \tag{6.61}$$

Also it can been seen that , for

$$h_1 < \min(H_1,H_2,H_3) \qquad i=1(1)N-1 \tag{6.62}$$

M is irreducibly diagonally dominant provided $\lambda \in (1/\sqrt{3},\ 1]$.

$$\text{where } H_1 = \frac{u_{i-\lambda}u_{i-1}\lambda(\lambda^2-1) + 6\varepsilon\sigma_i^3 v_{i-\lambda}}{\sigma_1(2\varepsilon\sigma_i\lambda u_{i-\lambda} - v_{i-\lambda}u_{i-1})}$$

$$H_2 = \frac{B_{2i}u_{i+\lambda}u_{i+1}\lambda(1-\lambda^2) - 6\varepsilon B_{2i}v_{i+\lambda} - 3B_{2i}v_{i-\lambda}u_{i+1}(\lambda^2 - 1/3)}{B_{2i}\lambda u_{i+\lambda}}$$

$$H_3 = \frac{[(1-\lambda)(B_{0i}u_{i-\lambda}+B_{2i}u_{i+\lambda})-B_{2i}v_{i+\lambda}]6\varepsilon\sigma_i^2+3(1/3-(1-\lambda)^2)\sigma_i^2 B_{2i}v_{i+\lambda}u_i}{(1-\lambda)[(1-\lambda^2)-1](B_{0i}u_{i-\lambda}u_i - \sigma_i^2 B_{2i}u_{i+\lambda}u_i)}$$

Let $S_i = \sum_{k=1}^{N-1} m_{ik}$, $\quad i=1(1)N-1$ (6.63)

For h_i satisfying (6.62), we have

$$S_i \geq 0 \tag{6.64}$$

and $S_i = h_i^2 G_i + O(h_i^3)$ (6.65)

where $G_i = B_{0i}u_{i-\lambda} + B_{1i}u_i + B_{2i}u_{i+\lambda} > 0$

Also, from the theory of matrices, we have

$$\sum_{i=1}^{N-1} m_{ji}^{-1} S_i = 1 \qquad j = 1(1)N-1 \tag{6.66}$$

where m_{ji}^{-1} is $(j,i)^{th}$ element of M^{-1}.

With the help of Eqs.(6.63)-(6.65) and for h_i satisfying (6.62), we have

$$\sum_{i=1}^{N-1} m_{ji}^{-1} \leq \frac{1}{\min\limits_{1\leq i\leq N-1} S_i} \leq \frac{1}{h_{i0}^2\, G_{i0}}$$

for some i_0 between 1 & N-1 (6.67)

From (6.60) we have

$$e_j = \sum_{i=1}^{N-1} m_{ji}^{-1}\, \bar{T}(h_i) \qquad j = 1(1)N-1 \tag{6.68}$$

and therefore

$$|e_j| \le \frac{h^5}{h_{i0}^2 G_{i0}} \qquad j = 1(1)N-1 \qquad (6.69)$$

where $h = \max\limits_{1 \le i \le N} h_i$

Thus we have

$$\|E\| = O(h^3), \qquad h = \max_{1 \le i \le N} h_i \qquad (6.70)$$

Remark 6.2 If $\sigma_i = 1$ $\forall i$ (uniform mesh), we have

$$T(h_i) = O(h_i^6),$$

Hence we have

$$\|E\| = O(h^4), \qquad h = \max_{1 \le i \le N} h_i \qquad (6.71)$$

6.5 NUMERICAL EXAMPLES :

Example 6.1: Convection - diffusion problem (Jain et. al.1984)

$$\varepsilon\, y'' = y' \qquad 0 < x < 1$$
$$y(0) = 1, \; y(1) = 0$$

The exact solution is

$$y(x) = \frac{1 - \exp(-(1-x)/\varepsilon)}{1 - \exp(-1/\varepsilon)}$$

The boundary layer exists near the right end, so we take $\sigma < 1$, with $h_0 = (1-\sigma)/(1-\sigma^N)$,$h_i = \sigma h_{i-1}$, i=1,N-1, where N is the number of mesh points in the interval. This ensures more mesh points in the boundary layer region.

Example 2: (Kevorkian & Cole, 1981)

$$\varepsilon\, y'' = \left(\frac{x}{2} - 1\right) y' + \frac{1}{2}\, y \qquad 0 < x < 1$$

$$y(0) = 0 \,, \quad y(1) = 1$$

We have taken Nayfey's uniformly valid approximation, (Nayfey, 1981) as exact solution for comparison, which is given as

$$y(x) = \frac{1}{2-x} - \frac{1}{2} \exp(-(x - \frac{x^2}{4})/\varepsilon)$$

The boundary layer exists near the left end, so we take $\sigma > 1$, with $h_1 = (\sigma - 1)/(\sigma^N - 1)$, $h_i = \sigma_i h_{i-1}$, $i=2,N$ .This ensures more mesh points near $x = 0$.

Example 6.3: (Bender & Orszag, 1978)

$$\varepsilon y'' = -2y' - e^y \qquad 0 < x < 1$$

$$y(0) = 0, \quad y(1) = 1$$

we have taken Bender & Orszag's uniformly valid approximation as exact solution for comparison, which is given by

$$y(x) = \log \frac{2}{1+x} - (\exp(-2x/\varepsilon))\log 2$$

The boundary layer region is at left end, so we follow the procedure as in example 2 for choosing σ.

The initial guess is taken as $u_0(x) = 0$ and stopping criteria is

$$|u_{n+1}(x_i) - u_n(x_i)| < 10^{-7} \quad \forall i$$

6.5. DISCUSSION :

We have implemented our method on three examples. Maximum error at the nodal points i.e., max $|y_i - y(x_i)|$ is tabulated in Tables 6.1-6.3 for different values of the parameters ε, N, λ, σ.
It is observed that the errors are comparatively less for values of σ other than unity, which shows that variable mesh with approximations at off-nodal poiunts gives better results.

Table 6.1

Maximum absolute error in Example 6.1

	N	σ	λ = 0.85	λ = 0.90	λ = 1.00
$\varepsilon=10^{-5}$	60	1.00	9.81746 E -1	9.81746 E -1	9.81746 E -1
		0.90	9.03708 E -2	9.25960 E -2	9.72461 E -2
		0.875	5.78031 E -4	6.12345 E -4	6.84009 E -4
	120	1.00	9.80875 E -1	9.80875 E -1	9.80875 E -1
		0.90	2.48247 E -5	2.66060 E -5	3.03009 E -5
		0.875	9.64444 E -6	1.03253 E -5	1.17449 E -5
$\varepsilon=10^{-8}$	150	1.00	9.91992 E -1	9.91992 E -1	9.91992 E -1
		0.90	9.99618 E -1	1.00000	1.00004
		0.875	9.99826 E -4	1.05306 E -3	1.17058 E -3
	300	1.00	9.99598 E -1	9.99598 E -1	9.99598 E -1
		0.90	2.48001 E -5	2.65643 E -5	3.02596 E -5
		0.875	9.53883 E -6	1.02153 E -5	1.16196 E -5
$\varepsilon=10^{-10}$	200	1.00	9.94998 E -1	9.94998 E -1	9.94998 E -1
		0.90	6.92775 E -4	7.28057 E -4	8.01135 E -4
		0.875	2.57257 E -5	2.71760 E -5	3.06257 E -5
	400	1.00	9.97499 E -1	9.97499 E -1	9.97499 E -1
		0.90	9.50779 E -5	9.92227 E -5	1.10196 E -5
		0.875	2.43328 E -5	2.72501 E -5	3.00868 E -5
$\varepsilon=10^{-12}$	300	1.00	9.96363 E -1	9.96363 E -1	9.96363 E -1
		0.90	7.80524 E -4	7.85154 E -4	7.86973 E -4
		0.875	2.09647 E -4	2.05907 E -4	2.12988 E -4
	600	1.00	9.98181 E -1	9.98181 E -1	9.98181 E -1
		0.90	7.77207 E -4	7.71633 E -4	7.71378 E -4
		0.875	4.42990 E -4	4.44399 E -4	4.93990 E -4

Table 6.2

Maximum absolute error in Example 6.2

	N	σ	λ = 0.85	0.90	1.00
$\varepsilon=10^{-5}$	60	1.00	4.93354 E -1	4.93205 E -1	4.92889 E -1
		1.10	4.92056 E -1	2.17676 E -1	4.80195 E -1
		1.15	3.33928 E -2	3.50604 E -2	4.91871 E -2
	120	1.00	4.92394 E -1	4.92176 E -1	4.91878 E -1
		1.10	3.23356 E -4	3.25330 E -4	3.30103 E -4
		1.15	6.90069 E -4	6.92035 E -4	6.96799 E -4
$\varepsilon=10^{-8}$	150	1.00	4.97903 E -1	4.97902 E -1	4.97901 E -1
		1.10	4.99997 E -1	4.99997 E -1	4.99993 E -1
		1.15	9.45442 E -4	9.73627 E -4	1.03259 E -3
	300	1.00	4.98946 E -1	4.98945 E -1	4.98942 E -1
		1.10	3.20503 E -4	3.20507 E -4	3.20512 E -4
		1.15	6.87343 E -4	6.87342 E -4	6.87346 E -4
$\varepsilon=10^{-10}$	200	1.00	4.98746 E -1	4.98746 E -1	4.98746 E -1
		1.10	4.99999 E -1	4.99999 E -1	4.99999 E -1
		1.15	3.44208 E -2	3.70527 E -2	3.21067 E -2
	400	1.00	4.99374 E -1	4.99374 E -1	4.99374 E -1
		1.10	2.80828 E -3	2.86294 E -3	2.92001 E -3
		1.15	3.20223 E -4	3.20960 E -4	3.20548 E -4
$\varepsilon=10^{-12}$	300	1.00	4.98997 E -1	4.99897 E -1	4.99897 E -1
		1.10	4.99999 E -1	4.99999 E -1	4.99999 E -1
		1.15	2.07622 E -1	1.62142 E -1	1.00568 E -1
	600	1.00	4.99499 E -1	4.98997 E -1	4.99499 E -1
		1.10	1.04976 E -3	1.07085 E -3	1.15266 E -3
		1.15	3.17444 E -4	2.85180 E -4	3.20599 E -4

Table 6.3

Maximum absolute error in Example 6.3

	N	σ	λ = 0.85	0.90	1.00
$\varepsilon=10^{-5}$	60	1.00	6.75894 E -1	6.75894 E -1	6.75894 E -1
		1.10	6.92318 E -1	6.92332 E -1	6.92324 E -1
		1.15	5.64459 E -2	5.65972 E -2	5.67671 E -2
	120	1.00	6.83299 E -1	6.82260 E -1	6.83234 E -1
		1.10	5.90123 E -2	4.90086 E -2	4.58723 E -2
		1.15	4.02583 E -2	3.76094 E -2	4.10463 E -4
$\varepsilon=10^{-8}$	150	1.00	6.84845 E -1	6.84845 E -1	6.84845 E -1
		1.10	6.93147 E -1	6.34251 E -1	6.97642 E -1
		1.15	4.85552 E -2	4.26693 E -2	4.81394 E -2
	300	1.00	6.68986 E -1	6.68986 E -1	6.68986 E -1
		1.10	9.10823 E -2	5.93021 E -2	4.75013 E -2
		1.15	4.27203 E -2	3.33572 E -2	4.18450 E -2
$\varepsilon=10^{-10}$	200	1.00	6.88160 E -1	6.88160 E -1	6.88160 E -1
		1.10	6.93144 E -1	6.93144 E -1	6.93144 E -1
		1.15	4.38710 E -2	3.02597 E -2	4.18482 E -2
	400	1.00	6.90650 E -1	6.90650 E -1	6.90650 E -1
		1.10	6.93114 E -1	6.93114 E -1	6.93114 E -1
		1.15	3.50238 E -2	3.23456 E -2	3.37818 E -2
$\varepsilon=10^{-12}$	300	1.00	6.93256 E -1	6.93256 E -1	6.93256 E -1
		1.10	6.84321 E -1	6.84321 E -1	6.84321 E -1
		1.15	4.34522 E -2	4.24642 E -2	4.42711 E -1
	600	1.00	6.93456 E -1	6.93456 E -1	6.93456 E -1
		1.10	6.83243 E -1	6.83243 E -1	6.83243 E -1
		1.15	4.27654 E -2	4.42113 E -2	4.39877 E -2

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